Astronomical Relativity 2017
Exam

January 3, 2018

Note: Mark every sheet you hand in with your name and student number, and number the sheets. The clarity of your solutions will factor significantly into the grades. It is not sufficient to write a few equations. You must define your variables, draw well-labeled figures where appropriate, and explain what you are doing. Use geometrized units ($c = G = 1$) throughout, unless specifically instructed otherwise. Note that the instructions are compulsory, for instance if you are instructed to skip mathematical details, lengthy mathematical calculations will result in no points.

1 Problem 1: Rotating black holes

In this problem you are asked to explain or describe a number of things related to rotating black holes. Do not use any equations. Use at most 10 lines for each separate item.

(a) (0.75 pt) What is frame dragging?

(b) (0.5 pt) What is meant by the ergosphere of a rotating black hole?

(c) (0.75 pt) Explain why non-equatorial orbits around a rotating black hole do not lie in a plane.

2 Problem 2: Falling into a non-rotating black hole

In their respective spacecrafts, Darth Vader and Luke Skywalker are engaged in mortal combat, near a non-rotating black hole of mass M. In Schwarzschild coordinates, the fight takes place at coordinate radius $10\,M$. Luke succeeds in damaging Darth Vader's spacecraft. As a result, Darth Vader falls radially into the black hole, starting from rest.

(a) (2.0 pt) Show that on Darth Vader's clock, the time elapsed before he hits the central singularity is $5\sqrt{5}\pi M$. *Hint:* Start by determining the value of the conserved quantity e (which arises from the Killing vector $\xi^\alpha = (1, 0, 0, 0)$ in Schwarzschild geometry), then express Darth Vader's 4-velocity in terms of e. It will also be useful to know the standard integral $\int_0^1 dx\, \left(\frac{1}{x} - 1\right)^{-\frac{1}{2}} = \pi/2$.

(b) (1.0 pt) How much time elapses on Luke Skywalker's clock, before Darth Vader hits the central singularity? Avoid complex calculations but explain your answer clearly.

3 Problem 3: Volume and lifetime of a closed universe

Consider a Universe that is described by a closed Friedman-Robert-Walker model. It can be shown that for such a Universe, the time t and scale factor a obey the relations

$$a(\eta) = A\,(1 - \cos\eta), \tag{3.1}$$
$$t(\eta) = A\,(\eta - \sin\eta), \tag{3.2}$$

which implictly express the evolution of the scale factor with time. In these expressions, A is a constant, and η is a parameter that runs from 0 to 2π.

(a) (1.0 pt) Show that the maximum spatial volume V_* of this universe is given by $V_* = 16\pi^2 A^3$ (*Hint:* in doing this calculation, the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ will be useful.).

(b) (1.0 pt) Show that the total lifetime T (i.e., the time between Big Bang and Big Crunch) for this universe, if it reaches a maximum volume of 2×10^{13} Mpc^3, is close to $T = 10^{11}$ years. (*Hint:* recall that 1 pc = 3.26 lightyears; don't worry about precise numerical values: if you don't have a pocket calculator with you, just make estimes using $\pi \sim 3$, $\pi^2 \sim 10$, etc.)

4 Problem 4: Universes without and with vacuum-energy

A Friedman-Robertson-Walker (FRW) Universe of any geometry (open, flat, or closed) obeys the Friedman equation

$$\dot{a}^2 - \frac{8\pi\rho}{3}a^2 = -k, \tag{4.1}$$

where a is the scale factor, and $k = -1$, 0 or 1, depending on the geometry. The total density ρ depends on the scale factor as

$$\rho = \rho_{\mathrm{crit}}\left(\Omega_{\mathrm{v}} + \frac{\Omega_{\mathrm{m}}}{a^3} + \frac{\Omega_{\mathrm{r}}}{a^4}\right), \tag{4.2}$$

where ρ_{crit} is the critical density and Ω_{m}, Ω_{r} and Ω_{v} are the density parameters in matter, radiation and vacuum energy.

(a) (0.75 pt) Consider a FRW Universe containing matter and radiation (in any combination), but no vacuum-energy. Show that for such a Universe, $a(t)$ always curves downwards as a function of time (i.e., the second derivative of $a(t)$ is always negative).

(b) (0.75 pt) Explain, using a diagram of $a(t)$, that this means that $1/H_0$ (where H_0 is the Hubble constant) is always larger than the age of such a Universe. Explain the things that you draw in the diagram and why you draw them that way.

(c) (0.5 pt) Show that the result in part (a) does *not* have to be valid if this Universe has non-zero vacuum energy.