Exam Quantum Computing

General remarks

1. This exam consists of 6 parts on 5 pages. Parts 1-5 are practical, part 6 is theory.
2. This is a 'closed book' exam; however, you are allowed to consult one A4 sheet with a collection of formulas (written on both sides).
3. You are allowed to use a basic calculator in the exam.
4. If you carry out a computation please write down how you derived the answer. If you only write down the answer and it is wrong, no points are given for the question.
5. Keep your answers concise and relevant.
6. Please write your answers down in a readable way.
7. You have to answer the questions in English.

May you do well!

1. Linear algebra

1.1. Compute the tensor product of the following two matrices: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

1.2. What is the main reason for introducing the tensor product in the context of quantum computing and quantum mechanics?

2. Basic quantum mechanics

2.1. Which pairs of expressions for quantum states represent the same state?

2.1.1. $|0\rangle$ and $-|0\rangle$

2.1.2. $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

2.2. Quantum physical systems evolve by unitary transformations. Show that this is one:

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$$

3. Dirac notation

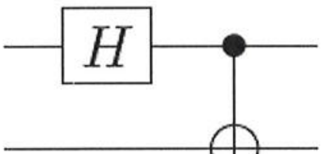
3.1. Compute the matrix representation of $P := |1\rangle\langle 1|$ by reverting to matrix representations of $|1\rangle$ and $\langle 1|$.

3.2. Show that $PP = P$ by using the matrix representation of P.

3.3. Show that $PP = P$ by using the Dirac notation.

4. Quantum gates, Bell states

4.1. Show that the following quantum circuit prepares the Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ on input $|00\rangle$: apply a Hadamard gate to the first Qbit followed by a CNOT with the fist Qbit as the control and the second Qbit as the target.

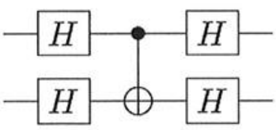


(The circuit outputs on input 10, 01, 11 the remainder of the Bell states)

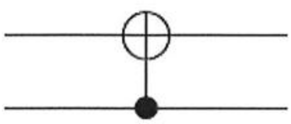
4.2. Show that the Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is an entangled state, i.e., cannot be written as the tensor product of two separate Qbits.

4.3. Show that if you apply the Hadamard gate to the inputs and outputs of a CNOT gate, the result is a CNOT gate with control and target qubits switched.

I.e., the following quantum circuit diagram



is equivalent to



4.4. Show why it is not possible to clone a quantum state, i.e., why we cannot construct unitary transformations which, for all possible actual states $|\psi\rangle$, are able to take this state $|\psi\rangle$ and create two copies of this state $|\psi\rangle\otimes|\psi\rangle$: $|\psi\rangle\otimes|0\rangle \rightarrow |\psi\rangle\otimes|\psi\rangle$. As a starting point, assume there exists such a unitary, which then must be able to clone $|0\rangle$ and $|1\rangle$:

$$U|0\rangle\otimes|0\rangle = |0\rangle\otimes|0\rangle, \quad U|1\rangle\otimes|0\rangle = |1\rangle\otimes|1\rangle$$

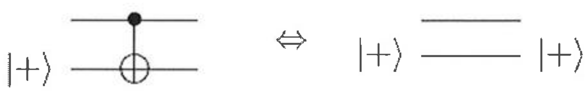
4.5. Is the no-cloning theorem quantum-only? What if we are dealing with a probabilistic device with the probability distribution $[\mathrm{p} \quad 1-\mathrm{p}]^{\mathrm{T}}$. Is it possible to turn this into two copies of the probability distribution

$$[p \quad 1-p]^{T}\otimes[p \quad 1-p]^{T}$$

for all possible probabilities p?

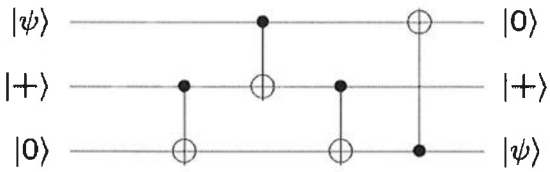
5. Quantum algorithms

5.1. In constructing the quantum teleportation algorithm we showed that

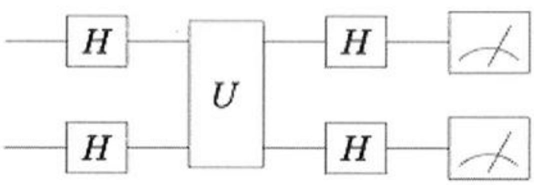


Show that this is true.

5.2. In the following intermediate circuit, at which we arrive when we construct the quantum teleportation algorithm from scratch, we can see a CNOT acting backwards. How can we get rid of it?



5.3. We showed that Deutsch's algorithm



allows us to distinguish whether the first set (f_1, f_2) or the second set (f_2, f_4) of the functions

$$f_1(x, y) = (x, y)$$
$$f_2(x, y) = (x, \bar{y})$$
$$f_3(x, y) = (x, x \oplus y)$$
$$f_4(x, y) = (x, x \oplus \bar{y})$$

has been queried only with one use of a quantum gate. This is a remarkable speedup compared to classical computers. How does it work? Explain and/or give mathematical definitions.

6. Theory

6.1. There is a remarkable harmony between mathematics and physics in the sense that both benefitted strongly from each other. Can the same be said from computer science and physics? Please argue (also have in mind classical information theory).

6.2. Why do we want to evolve our quantum physical systems via unitary transformations? What does unitary mean?

6.3. What is the difference between probability amplitudes in quantum physics and classical probabilities?