# Exam Algebraic Geometry 1

1 February 2022
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- Time allowed: 3 hours.
- You may consult the lecture notes, and your own notes for this course (including solutions of exercises). You may not consult any other sources (such as other textbooks, internet fora, etc.).
- Throughout, $k$ denotes an arbitrary **algebraically closed field** and all varieties are varieties over $k$.
- Every item (Exercise 1 i) etc.) has equal weight 1, except the final question (Exercise 4), which has weight 2.

**Exercise 1.** Consider the algebraic curve $C = \{((x:y),(s:t)) \mid x^2 s = y^2 t\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ and consider the rational function $f = \frac{xs}{yt} \in k(C)$.

i. Prove that $C$ is smooth and irreducible.

ii. Compute $\operatorname{div}(f)$. *Hint: use the four standard charts.*

iii. Compute $\operatorname{div}(df)$ and determine the genus of $C$.

**Exercise 2.** Suppose $X$ is a topological space and $\mathcal{F}$ is a sheaf of abelian groups on $X$. We define $I(\mathcal{F})(U) := \prod_{x \in U} \mathcal{F}_x$ for every open subset $U \subset X$, where $\mathcal{F}_x$ denotes the stalk of $\mathcal{F}$ at $x$.

i. Prove that $I(\mathcal{F})$, with the natural restriction maps, determines a sheaf of abelian groups on $X$.

ii. Find an injective morphism $\mathcal{F} \to I(\mathcal{F})$. *Hint: Recall that a morphism of sheaves of abelian groups $f : \mathcal{F} \to \mathcal{G}$ is injective when $f(U)$ is injective for every open subset $U \subset X$.*

iii. Show that for any sheaf of abelian groups $\mathcal{F}$ on $X$, there is a sheaf of abelian groups $\mathcal{G}$ on $X$ with an injective morphism $\mathcal{F} \to \mathcal{G}$ with the property that for each $V \subset U$, the restriction map $\mathcal{G}(U) \to \mathcal{G}(V)$ is surjective.

**Exercise 3.** Let $X = Z(I), Y = Z(J) \subset \mathbb{A}^n$ be affine algebraic varieties, where $I, J \subset k[x_1, \ldots, x_n]$ are *arbitrary* ideals.

i. Use the definitions to prove that $X \cup Y = Z(I \cap J)$ and $X \cap Y = Z(I + J)$ (for this part, you are not allowed to refer to the lecture notes). Here $I + J := \{f + g : f \in I \text{ and } g \in J\}$.

ii. Prove that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$. Deduce that $I(X \cup Y) = I(X) \cap I(Y)$.

iii. Is it always true that $I(X \cap Y) = I(X) + I(Y)$? If not, give a counter-example (over the ground field $k$, which is an algebraically closed field).

iv. Assume $\sqrt{I(X) + I(Y)} = I(X) + I(Y)$. Prove that there exists a short exact sequence of abelian groups

$$0 \to A(X \cup Y) \xrightarrow{\varphi} A(X) \oplus A(Y) \xrightarrow{\psi} A(X \cap Y) \to 0,$$

where $\varphi([f]) = ([f],[f])$ and $\psi([g],[h]) = [g-h]$. This means you have to prove that $\varphi$ is injective, $\psi$ is surjective, and $\ker(\psi) = \operatorname{im}(\varphi)$.

**Exercise 4.** Let $D$ be a divisor on an irreducible smooth projective curve $C$ of genus $g$. Assume that $\deg D = 2g - 2$ and $\dim H^1(C, \mathcal{O}_C(D)) > 0$. Prove that $\dim H^1(C, \mathcal{O}_C(D)) = 1$. *Hint: For a divisor $E$ on $C$, use the definition of $\mathcal{O}_C(E)$ to show the following: if $H^0(C, \mathcal{O}_C(E)) \neq 0$, then $E$ is linearly equivalent to an effective divisor.*