Analyse 3 NA, FINAL EXAM

*** Tuesday, January 17, 2017, 14.00 – 17.00 ***

Motivate each answer with a computation or explanation.
The maximum amount of points for this exam is 100.

No calculators!

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1. **(Holomorphic functions)** *[12 points]*

   Given is a function $v : \mathbb{R}^2 \to \mathbb{R}$ with $v(x,y) = 2xy + 5y$.

   (a) Show that v is harmonic.

   (b) Find a function $u : \mathbb{R}^2 \to \mathbb{R}$ such that the complex function $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic. Is u unique?

   (c) The function f from (b) is given as a function of x and y. Write it as a function of $z = x + iy$.

2. **(Residues, Laurent Series, Integrals)** *[28 points]*

   (a) Determine the principal part of the Laurent series around $z = 0$ of

   $$f(z) = \frac{(e^{iz} - 1)(1 - \cos(2z))}{z^4 \sinh(z)}.$$

   (b) Use residue calculus to compute the following definite integrals.

   $$(i) \int_0^{2\pi} \frac{1}{2 + \sin(\theta)}\, d\theta \qquad (ii) \int_0^{\infty} \frac{1}{(x^2+1)^2} \cos(\mu x)\, dx \quad (\mu \in \mathbb{R})$$

3. **(Ordinary differential equations)** *[20 points]*

   (a) Find the value of α for which the differential equation

   $$(xy^2 + \alpha x^2 y) + (x^3 + x^2 y)\frac{dy}{dx} = 0$$

   is exact. Give its general solution using that value of α.

   (b) Solve using Green's function the initial value problem

   $$y''(x) - 2y'(x) + y(x) = xe^x, \quad y(0) = 0, y'(0) = 0.$$

4. **(Fourier Analysis)** *[25 points]*

(a) **(Fourier Series)** Consider the periodic function

$$f(x) = x\cos(x)\,,\; x \in (-\pi, \pi)\,, \quad f(x+2\pi) = f(x)\,,\; x \in \mathbb{R}\,.$$

Compute its Fourier series.

Hint: You may use that $\cos(\alpha)\sin(\beta) = \frac{1}{2}\sin(\alpha+\beta) - \frac{1}{2}\sin(\alpha-\beta)$.

(b) **(Fourier Transform)** Consider the square pulse

$$g(x) = \begin{cases} 1\,, & x \in [-1,1] \\ 0\,, & x \in \mathbb{R}\setminus[-1,1] \end{cases} .$$

(i) Show that the Fourier transform of g is given by

$$\hat{g}(k) = \gamma\left(\frac{\sin(k)}{k}\right),$$

and compute γ.

(ii) Determine the integral $\int_{-\infty}^{\infty} |\hat{g}(k)|^2\, dk\,.$

(iii) Using (i), show that

$$\int_{-\infty}^{\infty} \frac{\sin(k/2)\cos(k)}{k}\, dk + \frac{\pi}{2} = \int_{-\infty}^{\infty} \frac{\sin(k/2)\sin(k)}{k^2}\, dk\,.$$

5. **(Second order ODEs with varying coefficients)** *[15 points]*

(a) Solve for $x > 1$ the initial value problem for the Euler differential equation

$$x^2 y''(x) - 2y(x) = 40x^7\,, \quad y(1) = 0\,, y'(1) = 2\,.$$

(b) Consider the differential equation

$$(1+z^2)y''(z) + 2zy'(z) + 4z^2 y(z) = 0\,,$$

for $y : \mathbb{C} \to \mathbb{C}$.

(i) What are the singular points of this ODE?

(ii) Determine a lower bound for the radius of convergence of series solutions expanded around $z = 0$.

End of Exam