Analyse 3 NA, EXAM 1

*** Monday, November 7, 2016, 14.00 – 16.00 ***

Motivate each answer with a computation or explanation.
The maximum amount of points for this exam is 100.

No calculators!

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1. **(Variation of parameters formula, Green's function)** *[25 points]*
   Consider the initial value problem
   $$5y''(x) - 15y'(x) + 10y(x) = 5\,e^x\,, \quad y(0) = 0\,, y'(0) = 0\,.$$
   (a) Compute the solution by the variation of parameters formula.
   (b) Find the Green's function.
   (c) Use the Green's function from (b) to confirm your findings from (a).

2. **(Fourier Series)** *[25 points]*
   Consider the function
   $$f(x) = \begin{cases} -1\,, & -2 \le x < -1 \\ x\,, & -1 \le x < +1 \\ +1\,, & +1 \le x < +2 \end{cases}$$
   and $f(x+4) = f(x), x \in \mathbb{R}$.
   (a) Show that the Fourier series of f is given by
   $$S_f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)\,, \quad b_n = \gamma\left(-\cos(n\pi) + \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)\,,$$
   with some $\gamma \in \mathbb{R}$ and determine γ.
   (b) Do we have $S_f(x) = f(x)$ for all $x \in \mathbb{R}$?
   (c) Derive from (a) that
   $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{2}{(2k+1)^2\pi}\,.$$
   (d) Compute the value of the series $\sum_{n=1}^{\infty} b_n^2$.

Please turn the page, there are more problems on the back!

3. **(Fourier Transform)** *[25 points]*

For $a > 0$ consider the function

$$f(x) = \begin{cases} -1\,, & -a \leq x < 0\,, \\ 1\,, & 0 \leq x < a\,, \\ 0\,, & \textit{otherwise}\,. \end{cases}$$

(a) Show that the Fourier transform of f is given by

$$\hat{f}(k) = A \cdot \frac{1 - \cos(ka)}{k}\,,$$

with some $A \in \mathbb{C}$ and determine A.

(b) Compute using (a) the value of the integral

$$\int_0^\infty \frac{1 - \cos(ka)}{k} \sin\left(\left(\frac{3a}{4}\right) k\right) dk\,.$$

(c) Compute the value of the integral

$$\int_{-\infty}^{\infty} \left(\frac{2k\sin(2k) - 1 + \cos(2k)}{k^2}\right) \cos(k)\, dk\,.$$

4. **(Power Series and the Frobenius Method)** *[25 points]*

Consider the ODE

$$zy''(z) + (1 - z)y'(z) - y(z) = 0\,,$$

which has $z = 0$ as regular singular point and a Frobenius series solution around $z = 0$.

(a) Determine the indicial equation.

(b) Compute one solution by the Frobenius method.

(c) Compute the Wronskian of the ODE.

(d) Combine (b) and (c) to derive an expression for the general solution of the ODE containing elementary functions. You may leave an integral in this expression.

End of Exam