Analyse 3 NA, EXAM 2

*** Monday, January 4, 2016, 14.00 – 16.00 ***

Motivate each answer with a computation or explanation.
The maximum amount of points for this exam is 50.

No calculators!

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1. **(First order ODEs)** *[10 points]*

   Consider the first order differential equation

   $$(3x^2y - y^2 + e^x) + (x^3 - 2xy)\,\frac{dy}{dx} = 0\,.$$

   (a) Is this equation exact?

   (b) Give an implicit relation for its solutions.

2. **(Second order ODEs)** *[16 points]*

   (a) Consider the differential equation

   $$y''(x) + 8y'(x) + 16y(x) = e^{-4x}\,.$$

   Solve the initial-value problem with

   $$y(0) = 0\,, y'(0) = 0\,,$$

   in two ways:

   (i) Using the Variation of Parameters formula and

   (ii) using the Green's function formalism.

   (b) Find a particular solution for the differential equation

   $$y''(x) + 4y'(x) + 3y(x) = 4 + \sum_{n=1}^{100} \frac{1}{n^2}\cos(nx)\,.$$

3. **(Fourier Analysis)** *[16 points]*

(a) Consider the 2π-periodic function $f(x) = e^x, x \in [-\pi, \pi), f(x + 2\pi) = f(x), x \in \mathbb{R}$.

(i) Compute its Fourier Series.

(ii) Using your result from (i), compute the value of the sum

$$\sum_{n=1}^{\infty} \frac{2}{1+n^2}.$$

(b) Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \begin{cases} \sin(x) & , \quad -\pi < x < \pi \\ 0 & , \quad otherwise \end{cases}.$$

Compute its Fourier transform.

4. **(Series Solutions of ODEs)** *[8 points]*

Consider for $y : \mathbb{C} \to \mathbb{C}$ the differential equation

$$y''(z) - \frac{1}{z}y'(z) + \left(1 + \frac{3}{4z^2}\right) y(z) = 0.$$

(a) Explain why $z = 0$ is a regular singular point.

(b) Compute two linearly independent solutions.

End of Exam