Analyse 3 NA, EXAM 1

*** Monday, November 6, 2017, 14.00 – 16.00 ***

Motivate each answer with a computation or explanation.
The maximum amount of points for this exam is 100.

No calculators!

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1. **(Variation of parameters formula, Green's function)** *[25 points]*
   Consider the initial value problem

$$\frac{1}{4}y''(x) - y(x) = 4e^{2x}\,, \quad y(0) = 0\,, y'(0) = 0\,.$$

   (a) Compute the solution by the variation of parameters formula.
   Solution: $y_1(x) = e^{2x}, y_2(x) = e^{-2x}, W(y_1, y_2)(x) = -4$ and the variation of parameters formula gives

$$y_p(x) = -y_1(x)\int \frac{y_2(x)16e^{2x}}{-4}dx + y_2(x)\int \frac{y_1(x)16e^{2x}}{-4}dx = 4xe^{2x} - e^{2x}\,.$$

   So the general solution is $y(x) = c_1e^{2x} + c_2e^{-2x} + 4xe^{2x}, c_1, c_2 \in \mathbb{R}$ and fitting the initial conditions finally gives $y(x) = e^{-2x} - e^{2x} + 4xe^{2x}$

   (b) Find the Green's function.
   Solution: The Green's function is of the form

$$G(x,z) = \begin{cases} A(z)e^{2x} + B(z)e^{-2x}\,, & 0 \le x < z < \infty\,, \\ C(z)e^{2x} + D(z)e^{-2x}\,, & 0 \le z < x < \infty\,. \end{cases}$$

   Fitting the initial conditions $G(0,z) = 0$, $\frac{d}{dx}G(0,z) = 0$ gives $A(z) = B(z) = 0$. Enforcing continuity at $x = z$ gives $0 = C(z)e^{2z} + D(z)e^{-2z}$, so $D(z) = -C(z)e^{4z}$, while using the jump condition at $x = z$ gives $4C(z)e^{-2z} = 1$ (for the ODE multiplied by 4), so $C(z) = \frac{1}{4}e^{-2z}$ and, finally, $D(z) = -\frac{1}{4}e^{2z}$. In summary,

$$G(x,z) = \begin{cases} 0\,, & 0 \le x < z < \infty\,, \\ \frac{1}{4}e^{2(x-z)} - \frac{1}{4}e^{-2(x-z)}\,, & 0 \le z < x < \infty\,. \end{cases}$$

   (c) Use the Green's function from (b) to confirm your findings from (a).
   Solution:

$$y(x) = \int_0^\infty G(x,z)16e^{2z}\,dz = \int_0^x \left(\frac{1}{4}e^{2(x-z)} - \frac{1}{4}e^{-2(x-z)}\right)16e^{2z}\,dz = e^{-2x} - e^{2x} + 4xe^{2x}\,.$$

2. **(Fourier Series)** *[25 points]*

For $0 < \alpha < \pi$ consider the function $f(x) = \sin(x), |x| < \alpha, f(x + 2\alpha) = f(x), x \in \mathbb{R}$.

(a) Sketch $f(x)$ in the range $-3\alpha < x < 3\alpha$.

(b) Show that the Fourier series of f is given by

$$S_f(x) = \sum_{n=1}^{\infty} (-1)^n \frac{\gamma n \sin(\alpha)}{\alpha^2 - n^2\pi^2} \sin\left(\frac{n\pi x}{\alpha}\right),$$

with some $\gamma \in \mathbb{R}$ and determine γ.

Solution: We have $L = \alpha$ and f an odd function, so $S_f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\alpha}\right)$ with

$$b_n = \frac{2}{\alpha}\int_0^{\alpha} \sin(x) \sin\left(\frac{n\pi x}{\alpha}\right) dx = (-1)^n \frac{2\pi n \sin(\alpha)}{\alpha^2 - n^2\pi^2}, \qquad \text{hence} \quad \gamma = 2\pi .$$

(c) Do we have $S_f(x) = f(x)$ for all $x \in \mathbb{R}$?

Solution: Since f is continuous for all $x \in \mathbb{R}$ with $x \neq k\alpha, k \in \mathbb{Z}$, we have $S_f(x) = f(x)$ there. At the jump discontinuities we have $S_f(k\alpha) = \frac{1}{2}(f([k\alpha]^+) + f([k\alpha]^-))$.

(d) What happens for the Fourier series S_f when $\alpha \to \pi$?

Solution: We have $b_n \to 0, n \geq 2$, and $b_1 \to 1$, so $S_f(x) = \sin(x)$.

(d) Derive from (a) the value of the series $\sum_{n=1}^{\infty} \frac{n^2}{(1-4n^2)^2}$.

Solution: Let $\alpha = \frac{\pi}{2}$. Then $b_n^2 = \frac{64}{\pi^2}\left(\frac{n^2}{(1-4n^2)^2}\right)$, so since by Parseval's identity we have $\sum_{n=1}^{\infty} b_n^2 = \frac{1}{\pi/2}\int_{-\pi/2}^{\pi/2} \sin(x)^2 dx = 1$ we have that the value of the series is $\frac{\pi^2}{64}$.

3. **(Fourier Transform)** *[25 points]*

Consider the function

$$f(x) = \begin{cases} x\, e^{-x}\,, & x > 0\,, \\ 0\,, & x \le 0\,. \end{cases}$$

(a) Show that the Fourier transform of f is given by

$$\hat{f}(k) = A\left(\frac{B+ik}{B^2+k^2}\right)^2\,,$$

with some $A, B \in \mathbb{R}$ and determine A and B.

Solution:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}\left(\frac{-1+ik}{1+k^2}\right)^2$$

therefore, $A = \frac{1}{\sqrt{2\pi}}$ and $B = -1$

(b) Compute the Fourier transform of f'.

Solution: $\mathcal{F}(f')(k) = (ik)\hat{f}(k)$.

(c) Using the previous calculations compute the value of the integral

$$\int_{-\infty}^{\infty} \frac{k^2}{k^4+2k^2+1}\,dk\,.$$

Solution: Combining (a) and (b) we have that

$$\frac{1}{2} = \frac{f'(0^+)+f'(0^-)}{2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(ik)\hat{f}(k)e^{ik\cdot 0}\,dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2k^2}{k^4+2k^2+1}\,dk$$

where we used that odd integrands vanish after integration. So the value is $\frac{\pi}{2}$.

(d) Consider the inhomogeneous ODE

$$-u''(x) + u(x) = (f * f)(x)\,,$$

which is forced by the convolution $f * f$. Compute the Fourier transform $\hat{u} = \mathcal{F}\{u\}$ of the solution u in terms of $\hat{f} = \mathcal{F}\{f\}$ and use this to express the solution u in terms of $\hat{f}$.

Solution: Fourier transformation of the ODE gives

$$(k^2+1)\hat{u}(k) = \frac{1}{\sqrt{2\pi}}\hat{f}(k)^2\,,$$

so

$$\hat{u}(k) = \frac{\hat{f}(k)^2}{\sqrt{2\pi}(k^2+1)}\,,$$

and, hence, the solution is of the form

$$u(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}(k)e^{ikx}\,dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\hat{f}(k)^2}{(k^2+1)}e^{ikx}\,dk\,.$$

4. **(Power Series and the Frobenius Method)** *[25 points]*
Consider for some $m \in \mathbb{N}$ the Chebyshev equation

$$(1-z^2)y''(z) - zy'(z) + m^2 y(z) = 0\,.$$

(a) Give all ordinary and singular points of this ODE.
Solution: $z = \pm 1$ are singular points, $z \in \mathbb{R} \setminus \{\pm 1\}$ are ordinary points

(b) Use a series ansatz around the point $z = 0$ and give the corresponding recurrence relation.
Solution: Since $z = 0$ is an ordinary point we can use the power series ansatz $y(z) = \sum_{n=0}^{\infty} a_n z^n$ which gives the recurrence relation

$$a_{n+2} = \frac{n^2 - m^2}{(n+2)(n+1)} a_n\,, \quad n \in \mathbb{N}_0$$

(c) Let $m = 2$. Show that there is a polynomial solution with $y(0) = 1, y'(0) = 0$.
Solution: Since $y(0) = 1, y'(0) = 0$ this fixes $a_0 = 1, a_1 = 0$ and by the recurrence relation we get $a_2 = \frac{-2^2}{2} = -2$ and $a_n = 0, n \geq 2$. Hence, we get the polynomial solution $y(z) = 1 - 2z^2$.

(d) Use your computations from (b) to determine two linearly independent solutions without solving the recurrence relation.
Solution: We can first set $a_0 = 1$ and $a_1 = 0$, which gives that all coefficients with odd index vanish and so one solution is

$$y_1(z) = \sum_{n \in \mathbb{N}_{even}} a_n z^n\,.$$

Furthermore, setting $a_0 = 0$ and $a_1 = 1$ gives that all coefficients with even index vanish and so another solution is

$$y_2(z) = \sum_{n \in \mathbb{N}_{odd}} a_n z^n\,.$$

Since $W(y_1, y_2)(0) = 1$ we have that the Wronskian never vanishes, so $\{y_1, y_2\}$ is a fundamental system.

End of Exam