**Analyse 3 NA, EXAM 1**

*** Monday, November 7, 2016, 14.00 – 16.00 ***

**Motivate each answer with a computation or explanation.**
**The maximum amount of points for this exam is 100.**

**No calculators!**

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1. **(Variation of parameters formula, Green's function)** *[25 points]*
Consider the initial value problem

$$5y''(x) - 15y'(x) + 10y(x) = 5\,e^{x}\,, \quad y(0) = 0\,, y'(0) = 0\,.$$

(a) Compute the solution by the variation of parameters formula.

(b) Find the Green's function.

(c) Use the Green's function from (b) to confirm your findings from (a).

(a) $y''(x) - 3y'(x) + 2y(x) = 0$ has char. eq. $\lambda^2 - 3\lambda + 2 = 0$ and so, $y_1(x) = e^{2x}, y_2(x) = e^{x}$ and $W(y_1, y_2)(x) = -e^{3x}$. The variation of parameters formula gives the special solution

$$y_p(x) = -e^{2x}\int \frac{e^{x}e^{x}}{-e^{3x}}dx + e^{x}\int \frac{e^{2x}e^{x}}{-e^{3x}}dx = -e^{x} - xe^{x}\,,$$

so the general solution is given by $y(x) = C_1e^{2x} + C_2e^{x} - xe^{x}$. Fitting initial conditions gives $C_1 = 1, C_2 = -1$, so $y(x) = e^{2x} - (x+1)e^{x}$.

(b) The Green's function is of the form

$$G(x,z) = \begin{cases} A(z)e^{2x} + B(z)e^{x}\,, & 0 \le x < z\,, \\ C(z)e^{2x} + D(z)e^{x}\,, & 0 \le z < x\,. \end{cases}$$

Fitting the initial conditions $G(0,z) = 0$, $\frac{d}{dx}G(0,z) = 0$ gives $A(z) = B(z) = 0$. Enforcing continuity at $x = z$ gives $0 = C(z)e^{2z} + D(z)e^{z}$, so $D(z) = -C(z)e^{z}$, while using the jump condition at $x = z$ gives $2C(z)e^{2z} - C(z)e^{z}e^{z} = 1$ (for the ODE devided by 5), so $C(z) = e^{-2z}$ and, finally, $D(z) = -e^{-z}$. In summary,

$$G(x,z) = \begin{cases} 0\,, & 0 \le x < z\,, \\ e^{2(x-z)} - e^{x-z}\,, & 0 \le z < x\,. \end{cases}$$

(c) The solution to the initial value problem (with ODE devided by 5) is given by

$$y(x) = \int_0^{\infty} G(x,z)e^{z}\,dz = \int_0^{x} \left(e^{2(x-z)} - e^{x-z}\right)e^{z}\,dz = e^{2x} - (x+1)e^{x}\,.$$

2. **(Fourier Series)** *[25 points]*

Consider the function

$$f(x) = \begin{cases} -1\,, & -2 \le x < -1 \\ x\,, & -1 \le x < +1 \\ +1\,, & +1 \le x < +2 \end{cases}$$

and $f(x+4) = f(x), x \in \mathbb{R}$.

(a) Show that the Fourier series of $f$ is given by

$$S_f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right), \quad b_n = \gamma\left(-\cos(n\pi) + \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right),$$

with some $\gamma \in \mathbb{R}$ and determine $\gamma$.

(b) Do we have $S_f(x) = f(x)$ for all $x \in \mathbb{R}$?

(c) Derive from (a) that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{2}{(2k+1)^2\pi}.$$

(d) Compute the value of the series $\sum_{n=1}^{\infty} b_n^2$.

(a) We have $L = 2$ and $f$ an odd function, so $S_f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$ with

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx + \int_1^2 \sin\left(\frac{n\pi x}{2}\right)dx$$
$$= \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) - \left(\frac{2}{n\pi}\right)\cos(n\pi)\,, \qquad \text{hence} \quad \gamma = \frac{2}{n\pi}.$$

(b) $S_f(x) = f(x), x \in \mathbb{R} \setminus \{\pm 2, \pm 6, \pm 10, \ldots\}$, $S_f(x) = \frac{f(x+)+f(x^-)}{2}, x \in \{\pm 2, \pm 6, \pm 10, \ldots\}$

(c) $f$ is continuous at $x = 1$, so

$$1 = f(1) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(-\cos(n\pi) + \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)\sin\left(\frac{n\pi}{2}\right)$$

Since $\sin\left(\frac{n\pi}{2}\right)^2 = 1, n = 2k+1, \sin\left(\frac{n\pi}{2}\right) = 0, n = 2k, k \in \mathbb{N}_0$ and $-\cos((2k+1)\pi)\sin\left(\frac{(2k+1)\pi}{2}\right) = (-1)^k$ we get

$$1 = \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi}\left((-1)^k + \frac{2}{(2k+1)\pi}\right)$$

(d) By Parseval's identity we have $\sum_{n=1}^{\infty} b_n^2 = \frac{1}{2}\int_{-2}^{2} f(x)^2 dx = \int_0^1 x^2 dx + \int_1^2 dx = \frac{4}{3}$.

3. **(Fourier Transform)** *[25 points]*

For $a > 0$ consider the function

$$f(x) = \begin{cases} -1\,, & -a \leq x < 0\,, \\ 1\,, & 0 \leq x < a\,, \\ 0\,, & otherwise\,. \end{cases}$$

(a) Show that the Fourier transform of $f$ is given by

$$\hat{f}(k) = A \cdot \frac{1 - \cos(ka)}{k}\,,$$

with some $A \in \mathbb{C}$ and determine $A$.

(b) Compute using (a) the value of the integral

$$\int_0^\infty \frac{1 - \cos(ka)}{k} \sin\left(\left(\frac{3a}{4}\right) k\right) dk\,.$$

(c) Compute the value of the integral

$$\int_{-\infty}^\infty \left(\frac{2k\sin(2k) - 1 + \cos(2k)}{k^2}\right) \cos(k)\, dk\,.$$

(a) $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^0 (-1) e^{-ikx} dx + \frac{1}{\sqrt{2\pi}} \int_0^a (+1) e^{-ikx} dx = \underbrace{-i\sqrt{\frac{2}{\pi}}}_{=A} \left(\frac{1-\cos(ka)}{k}\right)$

(b) $1 = f(3a/4) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \hat{f}(k) e^{ik(3a/4)} dk = \frac{2}{\pi} \int_0^\infty \left(\frac{1-\cos(ka)}{k}\right) \sin\left(\frac{3ak}{4}\right) dk$ , so value is $\frac{\pi}{2}$.

(c) Set $a = 2$. From $xf(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty i\hat{f}'(k) e^{ikx} dk$ and $\left(\frac{1-\cos(2k)}{k}\right)' = \left(\frac{2k\sin(2k)-1+\cos(2k)}{k^2}\right)$ which is an even function, one gets

$$1 = 1 \cdot 1 = 1 \cdot f(1) = \frac{1}{\sqrt{2\pi}} i \left(-i\sqrt{\frac{2}{\pi}}\right) \int_{-\infty}^\infty \left(\frac{2k\sin(2k) - 1 + \cos(2k)}{k^2}\right) \cos(k)\, dk\,.$$

so the value is $\pi$.

4. **(Power Series and the Frobenius Method)** *[25 points]*

Consider the ODE

$$zy''(z) + (1-z)y'(z) - y(z) = 0\,,$$

which has $z = 0$ as regular singular point and a Frobenius series solution around $z = 0$.

(a) Determine the indicial equation.

(b) Compute one solution by the Frobenius method.

(c) Compute the Wronskian of the ODE.

(d) Combine (b) and (c) to derive an expression for the general solution of the ODE containing elementary functions. You may leave an integral in this expression.

(a) We have $s(z) = 1 - z, \tau(z) = -z$, so $\sigma(\sigma - 1) + \sigma = 0 \Leftrightarrow \sigma^2 = 0$

(b) $y(z) = \sum_{n=0}^{\infty} a_n z^n$ gives the recurrence relation $a_{n+1} = \frac{a_n}{n+1}$, so $a_n = \frac{a_0}{n!}$ and $y(z) = a_0 e^z$.

(c) The Wronskian is given by $W(y_1, y_2)(z) = Ce^{-\int \frac{1-z}{z} dz} = C\frac{e^z}{z}$,

(d) From (b) and (c) we can derive

$$y_1(z)y_2'(z) - y_1'(z)y_2(z) = e^z y_2'(z) - e^z y_2(z) = C\frac{e^z}{z} \Leftrightarrow y_2'(z) - y_2(z) = C\frac{1}{z},$$

which is solved by

$$y_2(z) = De^z + Ce^z \int \frac{e^{-z}}{z} dz$$

and, by choosing $D = 0, C = 1$ and $y_1(z) = e^z$, we can write the general solution as

$$y(z) = C_1 y_1(z) + C_2 y_2(z) = C_1 e^z + C_2 e^z \int \frac{e^{-z}}{z} dz\,.$$

## End of Exam