Hierbij een eerste aanzet voor de beantwoording van de vraagstukken van tentamen EM1 van 28 juni 2004:

Opgave 1: Griffiths 2.46

$$\mathbf{E} = -\nabla V = -A\frac{\partial}{\partial r}\left(\frac{e^{-\lambda r}}{r}\right)\hat{\mathbf{r}} = -A\left\{\frac{r(-\lambda)e^{-\lambda r} - e^{-\lambda r}}{r^2}\right\}\hat{\mathbf{r}} = \boxed{Ae^{-\lambda r}(1+\lambda r)\frac{\hat{\mathbf{r}}}{r^2}}.$$

$\rho = \epsilon_0\,\nabla\cdot\mathbf{E} = \epsilon_0 A\left\{e^{-\lambda r}(1+\lambda r)\,\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) + \frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla\left(e^{-\lambda r}(1+\lambda r)\right)\right\}$. But $\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^3(\mathbf{r})$ (Eq. 1.99), and $e^{-\lambda r}(1+\lambda r)\delta^3(\mathbf{r}) = \delta^3(\mathbf{r})$ (Eq. 1.88). Meanwhile,

$$\nabla\left(e^{-\lambda r}(1+\lambda r)\right) = \hat{\mathbf{r}}\frac{\partial}{\partial r}\left(e^{-\lambda r}(1+\lambda r)\right) = \hat{\mathbf{r}}\left\{-\lambda e^{-\lambda r}(1+\lambda r) + e^{-\lambda r}\lambda\right\} = \hat{\mathbf{r}}(-\lambda^2 r e^{-\lambda r}).$$

So $\frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla\left(e^{-\lambda r}(1+\lambda r)\right) = -\frac{\lambda^2}{r}e^{-\lambda r}$, and $\boxed{\rho = \epsilon_0 A\left[4\pi\delta^3(\mathbf{r}) - \frac{\lambda^2}{r}e^{-\lambda r}\right]}$.

$$Q = \int\rho\,d\tau = \epsilon_0 A\left\{4\pi\int\delta^3(\mathbf{r})\,d\tau - \lambda^2\int\frac{e^{-\lambda r}}{r}4\pi r^2 dr\right\} = \epsilon_0 A\left(4\pi - \lambda^2 4\pi\int_0^\infty r e^{-\lambda r}dr\right).$$

But $\int_0^\infty r e^{-\lambda r}dr = \frac{1}{\lambda^2}$, so $Q = 4\pi\epsilon_0 A\left(1 - \frac{\lambda^2}{\lambda^2}\right) = \boxed{\text{zero.}}$ !

Opmerking: Het is essentieel het elektrische veld als een vector te schrijven, en de divergentie correct te berekenen (zie productregel 5) met een deltafunctie. De totale lading kan ook bepaald worden door de wet van Gauss te gebruiken, en de limiet van een oneindige bolstraal te nemen. Ook dan vindt u Q = 0.

Opgave 2: Griffiths 4.9

Het is essentieel op te merken bij deel a, dat een dipool alleen een kracht ondervindt in een niet-uniform elektrisch veld, en het elektrische veld van de puntlading is niet-uniform. Alhoewel de krachten gelijk (in grootte) en tegengesteld zijn, mag u bij onderdeel (a) niet schrijven $\mathbf{F} = q\mathbf{E}_{\text{dip}}$ ; u moet Eq 4.5 gebruiken. Of u moet eerst (b) uitrekenen, en dan de derde wet van Newton aanhalen. En u moet opletten dat de vektor **r** is gedefinieerd van q naar **p**. En bespreken wat de richting van de kracht is.

(a) $\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}$ (Eq. 4.5); $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} = \frac{q}{4\pi\epsilon_0}\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}}$.

$$\begin{aligned}
F_x &= \left(p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z}\right)\frac{q}{4\pi\epsilon_0}\frac{x}{(x^2+y^2+z^2)^{3/2}} \\
&= \frac{q}{4\pi\epsilon_0}\left\{p_x\left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}x\frac{2x}{(x^2+y^2+z^2)^{5/2}}\right] + p_y\left[-\frac{3}{2}x\frac{2y}{(x^2+y^2+z^2)^{5/2}}\right]\right. \\
&\quad \left. + p_z\left[-\frac{3}{2}x\frac{2z}{(x^2+y^2+z^2)^{5/2}}\right]\right\} = \frac{q}{4\pi\epsilon_0}\left[\frac{p_x}{r^3} - \frac{3x}{r^5}(p_x x + p_y y + p_z z)\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{p}}{r^3} - \frac{3\mathbf{r}(\mathbf{p}\cdot\mathbf{r})}{r^5}\right]_x. \\
\mathbf{F} &= \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\left[\mathbf{p} - 3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\right].}
\end{aligned}$$

(b) $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left\{3\left[\mathbf{p}\cdot(-\hat{\mathbf{r}})\right](-\hat{\mathbf{r}}) - \mathbf{p}\right\} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right]$. (This is from Eq. 3.104; the minus signs are because **r** points *toward* **p**, in this problem.)

$$\mathbf{F} = q\mathbf{E} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right].}$$

[Note that the forces are equal and opposite, as you would expect from Newton's third law.]

Opgave 3: Griffiths 4.15

(a) $\rho_b = -\nabla\cdot\mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2}$; $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \left\{\begin{array}{ll} +\mathbf{P}\cdot\hat{\mathbf{r}} = k/b & (\text{at } r=b), \\ -\mathbf{P}\cdot\hat{\mathbf{r}} = -k/a & (\text{at } r=a). \end{array}\right\}$

Gauss's law $\Rightarrow \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{r^2}\hat{\mathbf{r}}$. For $r<a$, $Q_{\text{enc}}=0$, so $\boxed{\mathbf{E}=0.}$ For $r>b$, $Q_{\text{enc}}=0$ (Prob. 4.14), so $\boxed{\mathbf{E}=0.}$ For $a<r<b$, $Q_{\text{enc}} = \left(\frac{-k}{a}\right)(4\pi a^2) + \int_a^r \left(\frac{-k}{\bar{r}^2}\right)4\pi\bar{r}^2 d\bar{r} = -4\pi ka - 4\pi k(r-a) = -4\pi kr$; so $\boxed{\mathbf{E} = -(k/\epsilon_0 r)\,\hat{\mathbf{r}}.}$

(b) $\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} = 0 \Rightarrow \mathbf{D} = 0$ everywhere. $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = 0 \Rightarrow \mathbf{E} = (-1/\epsilon_0)\mathbf{P}$, so $\boxed{\mathbf{E} = 0 \text{ (for } r<a \text{ and } r>b);}$ $\boxed{\mathbf{E} = -(k/\epsilon_0 r)\,\hat{\mathbf{r}} \text{ (for } a<r<b).}$

Opgave 4: Zie Griffiths Example 5.9 (blz 227), en Example 5.12 (blz 238)

Opgave 5: Zie Griffiths Opgave 7.31 en 7.32

**Problem 7.31**

The displacement current density (Sect. 7.3.2) is $\mathbf{J}_d = \epsilon_0\frac{\partial\mathbf{E}}{\partial t} = \frac{I}{A} = \frac{I}{\pi a^2}\hat{\mathbf{z}}$. Drawing an "amperian loop" at radius $s$,

$$\oint \mathbf{B}\cdot d\mathbf{l} = B\cdot 2\pi s = \mu_0 I_{d_{\text{enc}}} = \mu_0\frac{I}{\pi a^2}\cdot\pi s^2 = \mu_0 I\frac{s^2}{a^2} \Rightarrow B = \frac{\mu_0 I s^2}{2\pi s a^2}; \quad \boxed{\mathbf{B} = \frac{\mu_0 I s}{2\pi a^2}\,\hat{\boldsymbol{\phi}}.}$$

**Problem 7.32**

(a) $\mathbf{E} = \frac{\sigma(t)}{\epsilon_0}\hat{\mathbf{z}}$; $\sigma(t) = \frac{Q(t)}{\pi a^2} = \frac{It}{\pi a^2}$; $\boxed{\frac{It}{\pi\epsilon_0 a^2}\hat{\mathbf{z}}.}$

(b) $I_{d_{\text{enc}}} = J_d\pi s^2 = \epsilon_0\frac{dE}{dt}\pi s^2 = \boxed{I\frac{s^2}{a^2}.}$ $\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{d_{\text{enc}}} \Rightarrow B\,2\pi s = \mu_0 I\frac{s^2}{a^2} \Rightarrow \mathbf{B} = \boxed{\frac{\mu_0 I}{2\pi a^2}s\,\hat{\boldsymbol{\phi}}.}$

(c) A surface current flows radially outward over the left plate; let $I(s)$ be the total current crossing a circle of radius $s$. The charge density (at time $t$) is

$$\sigma(t) = \frac{[I - I(s)]t}{\pi s^2}.$$

Since we are told this is independent of $s$, it must be that $I - I(s) = \beta s^2$, for some constant $\beta$. But $I(a) = 0$, so $\beta a^2 = I$, or $\beta = I/a^2$. Therefore $I(s) = I(1 - s^2/a^2)$.

$$B\,2\pi s = \mu_0 I_{\text{enc}} = \mu_0[I - I(s)] = \mu_0\frac{s^2}{a^2} \Rightarrow \boxed{\mathbf{B} = \frac{\mu_0}{2\pi a^2}s\,\hat{\boldsymbol{\phi}}.} \checkmark$$