Uitwerking Tentamen KMb Fall/Winter 2012

1. Moment of inertia of a triangle

(a) The center of mass is given by

$$\mathbf{r}_{CM} = \frac{1}{M} \int_0^h dz \int_{-\frac{d}{2}\left(1-\frac{z}{h}\right)}^{+\frac{d}{2}\left(1-\frac{z}{h}\right)} dx\, \sigma \mathbf{r}$$

For symmetry reasons we know that $\mathbf{r}_{CM}$ lies on the z-axis. We only need to calculate where:

$$z_{CM} = \frac{1}{M} \int_0^h dz \int_{-\frac{d}{2}\left(1-\frac{z}{h}\right)}^{+\frac{d}{2}\left(1-\frac{z}{h}\right)} dx\, \sigma z$$

We find

$$z_{CM} = \frac{\sigma}{M} \int_0^h dz\, z d \left(1 - \frac{z}{h}\right) = \frac{\sigma d}{M} \int_0^h dz \left(z - \frac{z^2}{h}\right)$$

from which follows

$$z_{CM} = \frac{\sigma d}{M} \left(\frac{h^2}{2} - \frac{h^3}{3h}\right) = \frac{\sigma d h^2}{6M}$$

One can also use the fact that $M = \sigma d h/2$ to simplify this to

$$z_{CM} = \frac{h}{3}$$

(2 points)

(b) The moment of inertia around the z-axis is defined as

$$I_z = \int_0^h dz \int_{-\frac{d}{2}\left(1-\frac{z}{h}\right)}^{+\frac{d}{2}\left(1-\frac{z}{h}\right)} dx\, \sigma x^2$$

As a first step we integrate over x:

$$I_z = \sigma \int_0^h dz\, \frac{x^3}{3}\bigg|_{-\frac{d}{2}\left(1-\frac{z}{h}\right)}^{+\frac{d}{2}\left(1-\frac{z}{h}\right)} = \sigma \frac{d^3}{12} \int_0^h dz \left(1 - \frac{z}{h}\right)^3$$

Substituting z by $u = 1 - z/h$ we find

$$I_z = \sigma \frac{d^3 h}{12} \int_0^1 du\, u^3 = \sigma \frac{d^3 h}{48}$$

(2 points)

(c) The kinetic energy is given by $T = \frac{1}{2} I_z \omega^2 = \sigma \frac{d^3 h}{96} \omega^2$, the impulsmoment has only a non-vanishing z-component, namely $L_z = I_z \omega = \sigma \frac{d^3 h}{48} \omega$. (2 points)

2. Rotation around a fixed axis

(a) There are two forces $m\omega^2 r$ acting on the axis at a distance d apart. This leads to a torque $2\,(d/2)\, m\omega^2 r = m\omega^2 rd$ along the y-axis. As the body rotates the direction of the torque rotates with the body. (2 points)

(b) One of the hoofdassen (say the 1-axis) is along a line going through the 2 mass points. The other two are identical can thus point in any direction perpendicular to the 1-axis. Let us choose the 2-axis to be pointing in the direction of the y-axis. One has $I_1 = 0$ and $I_2 = I_3 = ml^2$ with $l^2 = d^2 + r^2$. We introduce the angle θ between the x- and the 1-axis: $\tan\theta = r/d$ (or $\cos\theta = d/l$ and $\sin\theta = r/l$). Thus $\omega_1 = \omega\cos\theta$, $\omega_2 = 0$ and $\omega_3 = -\omega\sin\theta$. From Euler's equations we find (using that $\dot{\omega}_i = 0$): $N_1 = N_3 = 0$ and

$$N_2 = \omega_3 \omega_1 \left(I_1 - I_3\right) = \omega^2 m l^2 \cos\theta \sin\theta = m\omega^2 rd$$

(3 points)

3. The pendulum

(a) Lagrangian:

$$L = \frac{m}{2} l^2 \dot{\theta}^2 + mgl\cos\theta$$

Euler equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = ml^2 \ddot{\theta} + mgl\sin\theta = 0$$

leading to the equation of motion

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

(2 points)

(b) For small amplitudes $\sin\theta \approx \theta$ and we can approximate the equation of motion by $\ddot{\theta} = -(g/l)\,\theta$ which is solved e.g. by $\theta(t) = \theta_0 \sin\left(\sqrt{g/l}t\right)$. (2 points)

(c) The force is the sum of the centrifugal force $m\dot{\theta}^2 l$ and the component of the gravity force in the direction of the connecting line between the point mass and the origin, i.e. $mg\cos\theta$. (2 points)

(d) new Lagrangian with $g = r - l$:

$$L' = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta + \lambda\,(r-l)$$

leads to Lagrange equations:

$$\frac{d}{dt}\frac{\partial L'}{\partial\dot{\theta}} - \frac{\partial L'}{\partial\theta} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin\theta = 0$$

$$\frac{d}{dt}\frac{\partial L'}{\partial\dot{r}} - \frac{\partial L'}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta - \lambda = 0$$

$$\frac{d}{dt}\frac{\partial L'}{\partial\dot{\lambda}} - \frac{\partial L'}{\partial\lambda} = l - r = 0$$

From this follows that

$$\lambda = -ml\dot{\theta}^2 - mg\cos\theta$$

(1 point)

(e) The force on the connecting line is then given by

$$F_r = \lambda\frac{\partial g}{\partial\dot{r}} = \lambda = -ml\dot{\theta}^2 - mg\cos\theta$$

which coincides with what we predicted in (c). (1 point)