LA 2 NA. toets november 2014.

1.a $U^*U = |\alpha|^2 \begin{pmatrix} -i & -i \\ i & -i \end{pmatrix} \begin{pmatrix} i & -i \\ i & i \end{pmatrix} = |\alpha|^2 \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

U is unitair als $U^*U = I$ dus $2|\alpha|^2 = 1$; met $\alpha > 0$ geeft dit $\alpha = \frac{1}{2}\sqrt{2}$.

b. Kar. polynoom : $\begin{vmatrix} \frac{1}{2}i\sqrt{2} - \lambda & -\frac{1}{2}i\sqrt{2} \\ \frac{1}{2}i\sqrt{2} & \frac{1}{2}i\sqrt{2} - \lambda \end{vmatrix} = (\lambda - \frac{1}{2}i\sqrt{2})^2 - \frac{1}{2}$

dus $\lambda = \frac{1}{2}i\sqrt{2} \pm \frac{1}{2}\sqrt{2}$ zijn de eigenwaarden. De e.v. zijn te bepalen via $\frac{1}{2}i\sqrt{2}\,x_1 - \frac{1}{2}i\sqrt{2}\,x_2 = (\frac{1}{2}i\sqrt{2} \pm \frac{1}{2}\sqrt{2})x_1$ dit geeft $x_2 = \pm i x_1$. Een orthonormale basis van e.v. is $\left\{ \frac{1}{2}\sqrt{2}\begin{pmatrix} 1 \\ i \end{pmatrix}, \frac{1}{2}\sqrt{2}\begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}$. Dus is $U = VDV^*$ met

$V = \frac{1}{2}\sqrt{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ en $D = \begin{pmatrix} \frac{1}{2}i\sqrt{2} + \frac{1}{2}\sqrt{2} & 0 \\ 0 & \frac{1}{2}i\sqrt{2} - \frac{1}{2}\sqrt{2} \end{pmatrix}$.

2 a. Laat $\underline{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ met $\langle \underline{a}, \underline{c} \rangle = \langle \underline{b}, \underline{c} \rangle = 0$ zijn: dus

$2c_1 - 2ic_2 - c_3 = 0$, $-ic_2 - c_3 = 0$ dit geeft
$W^\perp = \text{span}\left\{ \begin{pmatrix} 1 \\ -2i \\ -2 \end{pmatrix} \right\}$.

b. Laat $\underline{A} = \underline{a} = \begin{pmatrix} 2 \\ 2i \\ -1 \end{pmatrix}$ zijn, we zoeken $\underline{B} \in W$ zo dat $\langle \underline{A}, \underline{B} \rangle = 0$:

$$\underline{B} = \underline{b} - \frac{\langle \underline{A}, \underline{b} \rangle}{\langle \underline{A}, \underline{A} \rangle} \underline{A} = \begin{pmatrix} 0 \\ i \\ -1 \end{pmatrix} - \frac{3}{9}\begin{pmatrix} 2 \\ 2i \\ -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -2 \\ i \\ -2 \end{pmatrix}$$

Een orthonormale basis is nu $\Big\{ \underbrace{\frac{1}{3}\begin{pmatrix} 2 \\ 2i \\ -1 \end{pmatrix}}_{=\underline{A}_1}, \underbrace{\frac{1}{3}\begin{pmatrix} -2 \\ i \\ -2 \end{pmatrix}}_{=\underline{B}_1} \Big\}$.

c. $P_W = \underline{A}_1\underline{A}_1^* + \underline{B}_1\underline{B}_1^* = \frac{1}{9}\begin{pmatrix} 2 \\ 2i \\ -1 \end{pmatrix}\begin{pmatrix} 2 & -2i & -1 \end{pmatrix} + \frac{1}{9}\begin{pmatrix} -2 \\ i \\ -2 \end{pmatrix}\begin{pmatrix} -2 & -i & -2 \end{pmatrix}$

$= \frac{1}{9}\begin{pmatrix} 8 & -2i & 2 \\ 2i & 5 & -4i \\ 2 & 4i & 5 \end{pmatrix}$

(check : $P_W \begin{pmatrix} 2 & 0 & 1 \\ 2i & i & -2i \\ -1 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 2i & i & 0 \\ -1 & -1 & 0 \end{pmatrix}$)

3 a. $T(A+B) = J(A+B)-(A+B)J = JA-AJ+JB-BJ$
$= \tau(A)+\tau(B)$

$\lambda \in \mathbb{R}: \quad T(\lambda A) = J(\lambda A)-(\lambda A)J = \lambda(JA-AJ) = \lambda T(A).$

b. Neem als basis $\mathcal{B} = \left\{ \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix} \right\}$.

$T\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix} - \begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}$

$T\begin{pmatrix}0&1\\0&0\end{pmatrix} = \quad --- \quad = \begin{pmatrix}0&-1\\-1&0\end{pmatrix}$

$T\begin{pmatrix}0&0\\1&0\end{pmatrix} = \quad --- \quad = \begin{pmatrix}-1&0\\0&1\end{pmatrix}$

$T\begin{pmatrix}0&0\\0&1\end{pmatrix} = \quad --- \quad = \begin{pmatrix}-1&0\\0&1\end{pmatrix}$

dus $T_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix}0&-1&-1&0\\1&0&0&-1\\1&0&0&-1\\0&1&1&0\end{pmatrix}$.

c. $\operatorname{Ker} \tau_{\mathcal{B}}^{\mathcal{B}} = \left\{ \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} : -x_2-x_3=0,\ x_1-x_4=0 \right\}$

$= \operatorname{span}\left\{ \begin{pmatrix}0\\-1\\+1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix} \right\}$ dus $\operatorname{Ker}(\tau) = \operatorname{span}\{J, I\} = \operatorname{span}\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&-1\\1&0\end{pmatrix}\right\}$

$\operatorname{Im} \tau_{\mathcal{B}}^{\mathcal{B}} = \operatorname{sp}\left\{ \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix} \right\}$ dus $\operatorname{Im}(\tau) = \operatorname{span}\left\{ \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}-1&0\\0&1\end{pmatrix} \right\}$.

d. $\langle \tau(A), B \rangle = \operatorname{tr}(T(A)^\tau B) = \operatorname{tr}((JA-AJ)^\tau B) =$

$\operatorname{tr}(A^\tau J^\tau B - J^\tau A^\tau B) \overset{*}{=} \operatorname{tr}(A^\tau(J^\tau B - BJ^\tau)) \overset{**}{=}$

$+\operatorname{tr}(A^\tau(-JB+BJ)) = \operatorname{tr}(A^\tau(-T(B))) \overset{\dagger}{=} -\operatorname{tr}(A^\tau \tau(B))$
$= -\langle A, \tau(B) \rangle$

$*$ wegens $\operatorname{tr}(XY) = \operatorname{tr}(YX)$
$**$ " $J^\tau = -J$
$\dagger$ " $\operatorname{tr}(-X) = -\operatorname{tr}(X)$.