Retake Examination Physics Experiments 2
Tuesday 14 May 2019, 10:00-13:00, in HL 106/9

No documents allowed. Keep answers short, in English or Dutch.
All sub-questions (a, b, c, …) are assigned the same number of points.

1. 1D Fourier transforms (12 points)

The Fourier Transform (FT) of a Gaussian function of frequency $G(\omega)=\exp\left(-\frac{\tau^2\omega^2}{2}\right)$ is a Gaussian function of time, $g(t)=\frac{1}{\tau}\exp\left(-\frac{t^2}{2\tau^2}\right)$.

a) Find the FT of the product of two Gaussians functions of frequency with different time constants τ_1 and τ_2.

b) Apply the convolution theorem to the product in question a) and deduce the convolution product of two Gaussian functions of time with different widths.

We define the 1/e-halfwidth w of a Gaussian as the value of the variable for which the function takes a value equal to the maximum value divided by e ($e = 2.718...$).

c) Express the 1/e-halfwidth w of the product function in frequency space with the 1/e-halfwidths w_1 and w_2 of the two composing functions.

d) Express the 1/e-halfwidth T of the FT in time space of the convolution product as a function of the 1/e-halfwidths, T_1 and T_2, of the components. Interpret this result qualitatively by sketching the product function of frequency and its FT as a function of time.

2. 2D Fourier transforms (12 points)

We look for the Fraunhofer diffraction pattern of a pair of rectangular parallel slits, a distance d apart (this distance is larger than the slit width, see Fig. 1) for monochromatic light with wavelength λ.

a) Describe the diffraction pattern of a single slit with length L and width l (Hint: the intensity distribution can be written as product of functions of the coordinates along and across the slit). Draw a qualitative sketch of the 2D intensity distribution of the diffracted wave.

b) Notice that the diffracting object is the convolution product of a single rectangular slit with a pair of Dirac delta functions a distance d apart. Deduce the 2D diffraction pattern of the pair of rectangular slits and represent its intensity qualitatively in a sketch.

3. Fourier transform of a signal (18 points)

A signal is given by the following function of time: $a(t) = A + B\cos(2\pi Ft)$, where the frequency F is unknown. To determine this modulation frequency, Bob records the signal over a long time T with a time resolution (or time step) Δt. Then, he calculates the FT of the signal by a fast FT (FFT) algorithm.

a) We first assume that $FT >> 1$ and that $F\Delta t << 1$. Describe the expected Fourier transform (consider only positive frequencies).

b) To test his method Bob now applies it to $b(t) = A + B\cos(2\pi f_0 t)$, with the high frequency $f_0 = 0.7/\Delta t$ and he repeats the measurement and FFT as done in a). Which (positive) frequencies does he now obtain after FFT in the interval between 0 and Δt^{-1}? Sketch qualitatively the FT he obtains.

c) With an adder circuit, Bob generates a signal $c(t) = A + B\cos(2\pi Ft) + C\cos(2\pi f_0 t)$ and applies the same FFT method as in a) and b). Describe qualitatively the new spectrum he obtains and sketch it.

4. Filters and OpAmp (18 points)

To filter out low-frequency noise from a signal, Alice needs a high-pass filter. She places two identical CR circuits in series (see Fig. 2).

a) Show that the transfer function of this system is given by:

$$H(s) = \left[1 + \frac{3}{CRs} + \frac{1}{(CRs)^2}\right]^{-1}.$$

Explain whether loading occurs in this filter.

b) To make a better filter, Alice now uses two identical differentiator circuits based on two ideal OpAmps, also mounted in series. Draw the branching scheme of this filter, and calculate its transfer function. Clarify why this is an improvement on the system of (a).

c) Draw Bode plots (modulus and phase) for the system of question (b).

5. Nonideal OpAmp (16 points)

A nonideal OpAmp is mounted as a voltage inverter with two identical resistors R of 1 kΩ. The OpAmp has a gain function defined by $V_{out} = G(\omega)(V_+ - V_-)$, which is equal to:

$$G(\omega) = \frac{G_0}{1 + i\omega\tau}$$

The Gain-BandWidth-Product is 1 MHz and τ = 1 s.

a) Calculate the deviation from ideal inversion due to the finite gain and operation at non-zero frequency.

b) Compare the deviation from ideal inversion at frequencies of 1 kHz and 100 kHz. What do you conclude?

6. Noise (24 points)

The cantilever of an atomic-force microscope (AFM) can be modeled as a harmonic oscillator with position $X(t)$, mass m, restoring force $-kX$, and damping force $-\beta\dot{X}$ (see Fig. 3). The AFM rests on a table which is subject to a broad spectrum of vibrations, leading to vertical movement $x(t)$ of the table.
Hint: in a non-Galilean reference frame subject to acceleration $\ddot{x}$ an inertial force $-m\ddot{x}$ appears applied to mass m. We of course neglect any influence of the light AFM cantilever on the heavy table.

a) Show that the transfer function relating the cantilever's position $X(t)$ to the table's vibration amplitude $x(t)$ is given by:

$$H(s) = s^2\left[s^2 + \frac{\beta}{m}s + \frac{k}{m}\right]^{-1}.$$

b) The table's position $x(t)$ fluctuates due to a pink $1/f$ noise with spectral density such that the standard deviation of the table's position $\left\langle \Delta x^2 \right\rangle_{pn} = \frac{A}{f}\Delta f$. Deduce the standard deviation of the cantilever's position $X(t)$ induced by this pink noise in spectral bandwidth Δf.

c) In addition to the table's fluctuation, the tip is subject to thermal noise arising from excitation by microscopic processes also responsible for friction. What is the frequency dependence of this thermal noise?
The Johnson-Nyquist *intensity* noise induced by a resistor is $4k_BT/R$. Use a mechanical-electrical analogy to derive the thermal mechanical *velocity* noise source acting on the cantilever. Because all measurements are done close to the resonance frequency f_0 of the cantilever, we can relate the thermal position noise $\left\langle \Delta X^2 \right\rangle_{th}$ to the thermal velocity noise by $\left\langle \Delta X^2 \right\rangle_{th} = f_0^{-2}\left\langle \Delta \dot{X}^2 \right\rangle_{th}$.

d) Compare the thermal noise $\left\langle \Delta X^2 \right\rangle_{th}$ and the pink noise $\left\langle \Delta X^2 \right\rangle_{pn}$ induced by the table in a narrow bandwidth at the resonance of the cantilever.
The spectral density of the pink noise is given by $A = 10^{-20}$ m^2; $\beta/m = 10^2$ s^{-1}, $f_0 = 100$ kHz, $m = 10^{-9}$ kg, and $k_BT = 4\times10^{-21}$ J.
Which noise source dominates?

FIGURES:

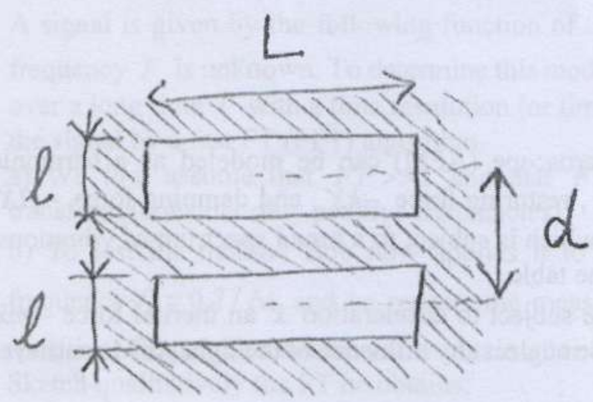

Figure 1: Scheme of the two rectangular slits of question 2.

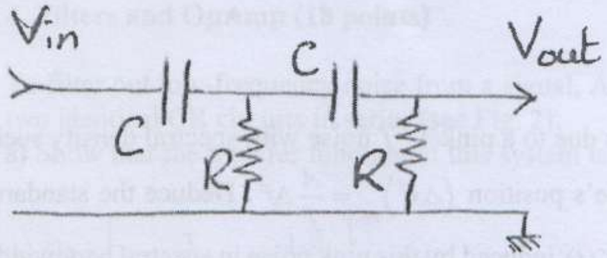

Figure 2: Two CR circuits mounted in series (question 4)

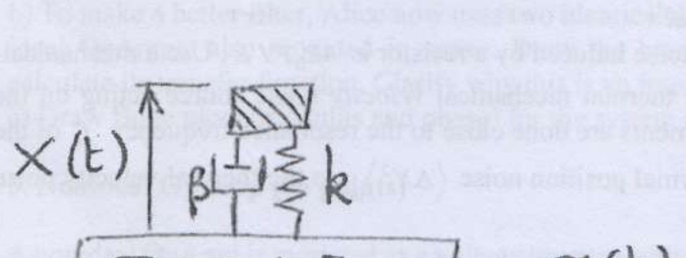

Figure 3: sketch of the AFM cantilever as a harmonic oscillator on a mobile table.