Exam Quantum Optics and Quantum Information

June 9, 2009

Problem 1.
We consider two modes of the radiation field, with the normalized mode functions $\vec{F}_1(\vec{r})$ and $\vec{F}_2(\vec{r})$. We consider the difference in expectation values for the two states $(|1,0\rangle \pm |0,1\rangle)/\sqrt{2}$, which are superpositions of one-photon states in these two modes.

a. Give an expression for the difference in expectation values $\langle \vec{E}_\perp^2(\vec{r})\rangle_+ - \langle \vec{E}_\perp^2(\vec{r})\rangle_-$ in these states.

b. Give an expression for the difference in expectation values $\langle \epsilon_0 \vec{E}_\perp(\vec{r}) \times \vec{B}(\vec{r})\rangle_+ - \langle \epsilon_0 \vec{E}_\perp(\vec{r}) \times \vec{B}(\vec{r})\rangle_-$ of the momentum density in these states.

c. Specialize these expressions in the case of two plane-wave modes $\vec{F}_\lambda(\vec{r}) = \vec{e}_\lambda \exp(i\vec{k}_\lambda \cdot \vec{r})/\sqrt{V}$, for $\lambda = 1,\, 2$.

Problem 2.
The normalized states with one photon in each one of two polarization modes A en B is given in the form (with horizontal or vertical linear photon polarization)

$$|\Psi\rangle = \frac{1}{\mathcal{N}}[4|HV\rangle + 5|VH\rangle + 3|VV\rangle] \;,$$

with $|HV\rangle = |H\rangle_A \otimes |V\rangle_B$, etc., and $\mathcal{N}$ a normalization constant.

a. Find the matrix form of the reduced density matrices $\hat{\rho}_A$ and $\hat{\rho}_B$.

b. Find the Schmidt decomposition of the state $|\Psi\rangle$.

c. Find the degree of entanglement of the state $|\Psi\rangle$.

d. The photon pair in the state $|\Psi\rangle$ is passed through a Bell-state analyser. Evaluate the probability for detecting each one of the four states $|\Psi^\pm\rangle = [|HH\rangle \pm |VV\rangle]\sqrt{2}$ and $|\Phi^\pm\rangle = [|HV\rangle \pm |VH\rangle]\sqrt{2}$.

Problem 3.
A lossless 50%-50% beam splitter has amplitudes for transmission and reflection $t = 1/\sqrt{2}$, $r = i/\sqrt{2}$.

a. What should be the ratio a/b between the classical input amplitudes a and b, so that the classical output amplitude $B = 0$?

b. Now we consider a one-photon quantum version of this same case, and we take the (normalized) one-photon input state $\alpha|1,0\rangle_{in} + \beta|0,1\rangle_{in}$, such that α/β is equal to the ratio a/b that was found in a. Express this state in terms of the output states, and check whether or not the output port B is empty.

c. One might guess that the two-photon quantum version of the same situation is the input state $\alpha|2,0\rangle_{in} + \beta|0,2\rangle_{in}$, with the same values of α and β. Express this state in terms of the output states, and give the probability distribution over the photon number in output port B.

d. What is the two-photon input state in which the output port B remains empty?

e. Give a product state of two coherent states in the input ports so that the output port B is in the vacuum state.

Problem 4.
The density matrix $\hat{\rho}_A$ of an open mode of the radiation field obeys the master equation (17.28). At the initial time $t = 0$ the mode is in the number state $|\psi_A(0)\rangle = |n\rangle$. The time-dependent density matrix of the mode can be decomposed in contributions corresponding to 0, 1, ..., decays, as in equation (17.19).

a. Evaluate the partial density matrix for zero decays $\hat{\tilde{\rho}}_0(t)$, which is a pure state.

b. Find an expression for the probability $P_0(t)$ that no decay has occurred, and for the waiting-time distribution $w(t)$ for the first decay.

c. Evaluate the partial density matrix for one decay $\hat{\tilde{\rho}}_1(t)$, and the probability $P_1(t)$ for precisely one decay.

d. Evaluate the partial density matrix $\hat{\tilde{\rho}}_2(t)$ corresponding to precisely two decays, and the corresponding probability $P_2(t)$.