Uitwerking tentamen QM1 12 februari 2013

1\. a. $\hat{H} = \dfrac{\hat{\vec{p}}^{\,2}}{2m} - \dfrac{e^2}{4\pi\varepsilon_0 \hat{r}}$

b. $n = 1, 2, 3, \ldots\ldots \qquad l = 0, 1, 2, \ldots, n-1 \qquad m = l, l-1, \ldots\ldots, -l$

c. $n\,; \quad E_n \sim \dfrac{1}{n^2}$

d. $\displaystyle\int_0^{2\pi}\int_0^{\pi} |\psi_{100}|^2 r^2 \sin\theta \, d\theta d\varphi = |\psi_{100}|^2 4\pi r^2 = 16\pi a^{-3} r^2 e^{-\frac{2r}{a}} \equiv F(r)$

$$\frac{dF}{dr} = 16\pi a^{-3}\left[2r\, e^{-\frac{2r}{a}} - \frac{2}{a} r^2 e^{-\frac{2r}{a}}\right] = 0 \rightarrow 2 - \frac{2r}{a} = 0 \rightarrow r = a$$

e. Voor waterstof $V = -\dfrac{e^2}{4\pi\varepsilon_0 r}$

Voor helium ion $V = -\dfrac{2e^2}{4\pi\varepsilon_0 r}$

} van $H \rightarrow He^+$
e^2 vervangen door $2e^2$

$a\,(\text{waterstof}) = \dfrac{4\pi\varepsilon_0 \hbar^2}{m e^2}$

$a\,(He^+) = \frac{1}{2}\, a\,(H)$

2\. a. $\hat{p}_x = \dfrac{\sqrt{m\hbar\omega}}{\sqrt{2}\, i}\,(\hat{a}_- - \hat{a}_+)$

$$\langle n|\hat{p}_x|n'\rangle = \frac{\sqrt{m\hbar\omega}}{\sqrt{2}\, i}\,\langle n|\hat{a}_- - \hat{a}_+|n'\rangle$$

$$= \frac{\sqrt{m\hbar\omega}}{\sqrt{2}\, i}\left\{\sqrt{n'}\,\delta_{n,n'-1} - \sqrt{n'+1}\,\delta_{n,n'+1}\right\}$$

dus alleen $\neq 0$ voor $n = n' \pm 1 \quad (n, n' = 0, 1, 2, \ldots\ldots)$

In matrixvorm

$$\frac{\sqrt{m\hbar\omega}}{\sqrt{2}\, i}\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & \sqrt{2} & & \phi \\ & -\sqrt{2} & 0 & \sqrt{3} & \\ & & & \ddots & \\ & \phi & & & \ddots \end{pmatrix}$$

b. Matrixrepresentatie in basis van eigenvectoren $\longrightarrow$ eigenwaarden op diagonaal

$$\langle n|\hat{H}|n'\rangle = E_n\,\delta_{nn'}$$

$$\begin{pmatrix} \frac{1}{2}\hbar\omega & & & \\ & \frac{3}{2}\hbar\omega & & \phi \\ & & \frac{5}{2}\hbar\omega & \\ \phi & & & \ddots \end{pmatrix}$$

c. $$\Psi(x,t) = \exp\left(\frac{-i\omega t}{2}\right)\sum_{n=0}^{\infty} c_n \psi_n \exp(-in\omega t)$$

met $c_0 = c_1 = c_3 = \frac{1}{\sqrt{3}}$ en andere $c_n = 0$

d.

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}_+ + \hat{a}_-)$$

$$\langle\Psi|\hat{x}|\Psi\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left\{\langle\Psi|\hat{a}_+|\Psi\rangle + \langle\Psi|\hat{a}_-|\Psi\rangle\right\}$$

$$\langle\Psi|\hat{a}_+|\Psi\rangle = \frac{1}{3}\exp(i\omega t)\,\langle\psi_1|\hat{a}_+|\psi_0\rangle = \frac{1}{3}\exp(i\omega t)$$

idem $\langle\Psi|\hat{a}_-|\Psi\rangle = \frac{1}{3}\exp(-i\omega t)$

$$\rightarrow \langle\Psi|\hat{x}|\Psi\rangle = \frac{2}{3}\sqrt{\frac{\hbar}{2m\omega}}\cos\omega t$$

$$\hat{p}_x = \frac{\sqrt{m\hbar\omega}}{\sqrt{2}\,i}(\hat{a}_- - \hat{a}_+)$$

idem $\longrightarrow \langle\psi|\hat{p}_x|\psi\rangle = -\frac{2}{3}\sqrt{\frac{m\hbar\omega}{2}}\sin\omega t$

3. a. $\hat{L}^2 Y_1^0 = 2\hbar^2 Y_1^0$

$\hat{L}^2 Y_1^1 = 2\hbar^2 Y_1^1$ } waarde $2\hbar^2$ met kans 1.

b. $\hat{S}_z \chi_+ = \frac{1}{2}\hbar\chi_+$

$\hat{S}_z \chi_- = -\frac{1}{2}\hbar\chi_-$ } waarden $\frac{1}{2}\hbar$ met kans $\frac{1}{3}$ en $-\frac{1}{2}\hbar$ met kans $\frac{2}{3}$

c.

kansdichtheid

$$\int_0^{2\pi}\int_0^{\pi}\frac{1}{3}R_{21}^2|Y_1^0|^2\sin\theta\,d\theta\,d\varphi = \frac{1}{3}R_{21}^2 = \frac{1}{72}a^{-5}r^2\exp\left(-\frac{r}{a}\right)$$

d. $\langle R_{21}(\sqrt{\tfrac{1}{3}}Y_1^0\chi_+ + \sqrt{\tfrac{2}{3}}Y_1^1\chi_-)|\hat{S}_x|R_{21}(\sqrt{\tfrac{1}{3}}Y_1^0\chi_+ + \sqrt{\tfrac{2}{3}}Y_1^1\chi_-)\rangle$

$= \langle R_{21}\sqrt{\tfrac{1}{3}}Y_1^0\chi_+|\tfrac{1}{2}(\hat{S}_+ + \hat{S}_-)|R_{21}\sqrt{\tfrac{1}{3}}Y_1^0\chi_+\rangle$

$+ \langle R_{21}\sqrt{\tfrac{2}{3}}Y_1^1\chi_-|\tfrac{1}{2}(\hat{S}_+ + \hat{S}_-)|R_{21}\sqrt{\tfrac{2}{3}}Y_1^1\chi_-\rangle = 0$