Uitwerking Toets QM 1

19 november 2002

1)
a. $\hat{H}\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$
b. $\exp\left(\frac{-iEt}{\hbar}\right)$
c. $\hat{H}\psi(x) = E\psi(x)$
d. $|\Psi(x,t)|^2 \neq f(t)$, dwz $\langle Q\rangle \neq f(t)$ voor alle $Q \neq Q(t)$.

2) a. $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$ voor $0 \leq x \leq a$.

$$\hat{H}\psi_n(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left(\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}\right) = -\frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)\frac{d}{dx}\left(\sqrt{\frac{2}{a}}\cos\frac{n\pi x}{a}\right)$$
$$= \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2\left(\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}\right) = \underbrace{n^2\frac{\pi^2\hbar^2}{2ma^2}}_{\text{eigenwaarde}}\ \psi_n(x)$$

b.

$$\langle\psi_m|\psi_n\rangle = \int_0^a \frac{2}{a}\sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}dx = \frac{1}{a}\int_0^a\left(\cos\frac{(m-n)\pi x}{a} - \cos\frac{(m+n)\pi x}{a}\right)dx$$

$$= \frac{1}{a}\left[\frac{a}{(m-n)\pi}\sin\frac{(m-n)\pi x}{a}\Bigg|_0^a - \frac{a}{(m+n)\pi}\sin\frac{(m+n)\pi x}{a}\Bigg|_0^a\right] = 0 \quad \text{voor } m \neq n$$

$$\langle\psi_n|\psi_n\rangle = \int_0^a \frac{2}{a}\sin^2\frac{n\pi x}{a}dx = \frac{1}{a}\int_0^a\left(1 - \cos\frac{2n\pi x}{a}\right)dx = 1$$

$\rightarrow \langle\psi_m|\psi_n\rangle = \delta_{mn}$
c.

$$\langle\psi_m|\hat{H}\psi_n\rangle = \langle\psi_m|n^2\frac{\pi^2\hbar^2}{2ma^2}\psi_n\rangle = n^2\frac{\pi^2\hbar^2}{2ma^2}\langle\psi_m|\psi_n\rangle = n^2\frac{\pi^2\hbar^2}{2ma^2}\delta_{mn}$$

Diagonale matrix met eigenwaarden op de diagonaal;
Correct, immers matrixrepresentatie van een operator in de basis van eigenfuncties is diagonaal met eigenwaarden op de diagonaal.
d. —

3) a. $V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$

$$\hat{x} = \frac{1}{\sqrt{2mi\omega}}\left(\hat{a}_+ - \hat{a}_-\right) \rightarrow \hat{x}^2 = \frac{-1}{2m\omega^2}\left(\hat{a}_+^2 - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+ + \hat{a}_-^2\right)$$

$$\langle\psi_n|\hat{V}(x)|\psi_n\rangle = -\frac{1}{4}\langle\psi_n|\hat{a}_+^2 - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+ + \hat{a}_-^2|\psi_n\rangle = -\frac{1}{4}\langle\psi_n| - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+|\psi_n\rangle$$

$$\left.\begin{array}{l} \hat{a}_+\hat{a}_-|\psi_n\rangle = \sqrt{n\hbar\omega}\hat{a}_+|\psi_{n-1}\rangle = n\hbar\omega|\psi_n\rangle \\ \hat{a}_-\hat{a}_+ = \sqrt{(n+1)\hbar\omega}\hat{a}_-|\psi_{n+1}\rangle = (n+1)\hbar\omega|\psi_n\rangle \end{array}\right\} (*)$$

$$\langle\psi_n|V|\psi_n\rangle = \frac{1}{4}\left(n\hbar\omega\langle\psi_n|\psi_n\rangle + (n+1)\hbar\omega\langle\psi_n|\psi_n\rangle\right) = \frac{1}{4}(2n+1)\hbar\omega$$

b. $\underbrace{[\hat{a}_+, \hat{a}_-]}|\psi_n\rangle = \left(\hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+\right)|\psi_n\rangle =^{(*)} \underbrace{-\hbar\omega}|\psi_n\rangle$

c.

$$\int_0^\infty |\psi(x,0)|^2 dx = 1 = |A|^2\langle\psi_0 + \psi_1|\psi_0 + \psi_1\rangle = 2|A|^2 \leftrightarrow A = \frac{1}{\sqrt{2}}$$

d.

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left\{\psi_0(x)\exp\left(\frac{-iE_0 t}{\hbar}\right) + \psi_1(x)\exp\left(\frac{-iE_1 t}{\hbar}\right)\right\}$$

e.

$$\langle\Psi|V|\Psi\rangle = \frac{1}{2}\langle\psi_0 e^{-iE_0 t/\hbar} + \psi_1 e^{-iE_1 t/\hbar}|V|\psi_0 e^{-iE_0 t/\hbar} + \psi_1 e^{-iE_1 t/\hbar}\rangle =$$

$$\frac{1}{2}\left\{\underbrace{\langle\psi_0|V|\psi_0\rangle}_{\neq f(t)} + \underbrace{\langle\psi_0|V|\psi_1\rangle}_{=0} e^{i(E_0-E_1)t/\hbar} + \underbrace{\langle\psi_1|V|\psi_0\rangle}_{=0} e^{-i(E_0-E_1)t/\hbar} + \underbrace{\langle\psi_1|V|\psi_1\rangle}_{\neq f(t)}\right\}$$

$\rightarrow \langle V\rangle \neq f(t)$ dwz behouden grootheid.