Uitwerking Toets QM 1

8 december 2003

1)
a. $\hat{H}(x,t)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$
b. $V \neq V(t)$
c. $\Psi(x,t) \equiv \varphi(x)f(t)$

$$\hat{H}(x)\varphi(x)f(t) = i\hbar\varphi(x)\frac{f(t)}{dt} \rightarrow \frac{1}{\varphi}\hat{H}\varphi = \frac{i\hbar}{f}\frac{df}{dt} \equiv E$$

$$\rightarrow \frac{df}{dt} = \frac{E}{i\hbar}f \rightarrow f \;\; \exp\left(\frac{-iEt}{\hbar}\right) \quad \text{met } E \text{ oplossing van } \hat{H}\varphi = E\varphi$$

2) a. $[\hat{H}, \hat{Q}] = 0 \rightarrow \frac{d}{dt}\langle Q\rangle = 0 \rightarrow \langle Q\rangle$ behouden.
b.

$$\left.\begin{array}{l} \frac{d}{dt}\langle p_x\rangle = \frac{i}{\hbar}\langle[\hat{H}, \hat{p}_x]\rangle \\ [\hat{H}, \hat{p}_x] = [\frac{\hat{p}_x^2}{2m} + \hat{V}, \hat{p}_x] = [\hat{V}, \hat{p}_x] = -\frac{\hbar}{i}\left(\frac{dV}{dx}\right) \end{array}\right\} \Rightarrow$$

$$[\hat{V}, \hat{p}_x]\psi = V\frac{\hbar}{i}\frac{d\psi}{dx} - \frac{\hbar}{i}(V\psi) = -\frac{\hbar}{i}\left(\frac{dV}{dx}\right)\psi$$

$$\Rightarrow \frac{d}{dt}\langle p_x\rangle = \langle -\left(\frac{dV}{dx}\right)\rangle$$

c. Verwachtingswaarden beschouwen als klassieke observabelen
geeft $\frac{d}{dt}p_x = -\frac{dV}{dx} = F_x$ d.w.z. 2^e wet van Newton.

3) a. $\hat{H} = -\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}$

$$\hat{H}\psi_n(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left(\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}\right) = \frac{\hbar^2}{2m}\frac{n^2\pi^2}{a^2}\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$$

$$= \frac{n^2\pi^2\hbar^2}{2ma^2}\psi_n(x) \longrightarrow \text{eigenwaarden } n^2\frac{\pi^2\hbar^2}{2ma^2} \quad (n = 1, 2, ...)$$

b. Deeltje in $\psi_1 : \psi_1(x) = \sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}$ voor $0 \leq x \leq a$
$\psi_1(x) = 0$ voor $a < x \leq 2a$
Eigentoestanden $\hat{H}$ na verplaatsen $\equiv \psi_n'(x) = \sqrt{\frac{1}{a}}\sin\frac{n\pi x}{2a}$ voor $0 \leq x \leq 2a$.

Iedere toestand, dus ook ψ_1 te schrijven als lin. comb. van $\psi'_n \quad (n=1,2,...) \longrightarrow$
$\psi_1 = \sum_n c_n \psi'_n \longrightarrow$ kans om deeltje in ψ'_1 aan te treffen $= |c_1|^2 = |\langle\psi'_1|\psi_1\rangle|^2$.

$$\langle\psi'_1|\psi\rangle = \int_0^{2a} \psi'_1 \psi_1 dx = \int_0^a \sqrt{\frac{1}{a}}\sqrt{\frac{2}{a}} \sin\frac{\pi x}{2a} \sin\frac{\pi x}{a} dx = \frac{4\sqrt{2}}{3\pi}$$

$|c_1|^2 = \frac{32}{9\pi^2} (\approx 0,36)$.

4) a.

$$\left.\begin{array}{l} \hat{a}_+\hat{a}_-\psi_n = \hat{a}_+\sqrt{n\hbar\omega}\psi_{n-1} = n\hbar\omega\psi_n \\ \hat{a}_-\hat{a}_+\psi_n = \hat{a}_+\sqrt{(n+1)\hbar\omega}\psi_{n+1} = (n+1)\hbar\omega\psi_n \end{array}\right\} [\hat{a}_+, \hat{a}_-] = -\hbar\omega$$

b.

$$\frac{1}{2}\hbar\omega \begin{pmatrix} 1 & & & \\ & 3 & & \\ & & 5 & \\ & & & \ddots \end{pmatrix}$$ matrixrepresentatie is diagonaal in de basis van eigenvectoren.

c. $\Psi(x,t) = \sum\limits_{n=0}^{\infty} c_n \psi_n(x) \exp\left(\frac{-iE_n t}{\hbar}\right)$.

d.

$$\langle\Psi(x,t)|\hat{Q}\Psi(x,t)\rangle = \sum_{n,m=0}^{\infty} c_n^* c_m \langle\psi_n|\hat{Q}\psi_m\rangle \exp\left(\frac{-i(E_m-E_n)t}{\hbar}\right) =$$

$$\{E_n - E_m = (n-m)\hbar\omega \equiv p\hbar\omega; \quad n,m \in [0,\infty] \to p \in [-\infty,\infty]\}$$

$$= \sum_{p=-\infty}^{+\infty} A_p \exp(ip\omega t) \quad \text{met} \quad A_p = \sum_{n=0}^{\infty} c_n^* c_{n-p} \langle\psi_n|Q\psi_{n-p}\rangle$$

e. $\hat{a}_+ - \hat{a}_- = \frac{1}{\sqrt{2m}}(2im\omega\hat{x})$

$$\langle\psi_n|\hat{x}|\psi_m\rangle = \frac{1}{i\sqrt{2m\omega}}\langle\psi_n|(\hat{a}_+ - \hat{a}_-)\psi_m\rangle = \frac{1}{i\sqrt{2m\omega}}\left(\sqrt{(m+1)\hbar\omega}\underbrace{\langle\psi_n|\psi_{m+1}\rangle}_{=\delta_{n,m+1}} - \sqrt{m\hbar\omega}\underbrace{\langle\psi_n|\psi_{m-1}\rangle}_{=\delta_{n,m-1}}\right)$$

$$A_p = \frac{\sqrt{\hbar}}{i\sqrt{2m\omega}}\sum_{n=0}^{\infty} c_n^* c_{n-p}\left(\sqrt{n-p+1}\delta_{n,n-p+1} - \sqrt{n-p}\delta_{n,n-p-1}\right)$$

$$\begin{array}{ll} A_1 = \frac{\sqrt{\hbar}}{i\sqrt{2m\omega}} \sum_{n=0}^{\infty} (c_n^* c_{n-1}\sqrt{n}) & \\ A_{-1} = \frac{\sqrt{\hbar}}{i\sqrt{2m\omega}} \sum_{n=0}^{\infty} \left(-c_n^* c_{n+1}\sqrt{n+1}\right) & A_p = 0 \forall p \neq \pm 1 \end{array}$$