Resit - RED - July 1, 2016 - 14:00-17:00

Problem 1. Time dilation [total: 2 points]

The average lifetime of a π meson in its own frame of reference is $2.6 \cdot 10^{-8}$ s. (This is its proper lifetime)

a. [0.5 point] If the π meson moves with speed 0.95c with respect to the Earth, what is its lifetime as measured by an observer at rest on Earth?

b. [0.5 points] What is the average distance it travels before decaying as measured by an observer at rest on Earth?

The half-life (i.e. the time required for the decaying quantity to fall to one half of its initial value) of a π meson at rest is $2.5 \cdot 10^{-8}$ s. A beam of π mesons is generated with velocity $v = \sqrt{4/5}\,c$ at a point d from a detector (as measured in the lab frame). Only half of the π mesons are observed to survive and reach the detector.

c. [1 point] Find the distance d to the detector.

Problem 2. Potentials and Fields of a moving magnetic dipole - [total: 3 points]

An ideal magnetic dipole is at rest at the origin in system $\mathcal{S}_0$. The components of the corresponding vector potential is

$$\boldsymbol{A}_0 = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r_0}}{r_0^3}, \tag{1}$$

while the electric potential V_0 is zero. [Where 0 here indicates the frame, not components]. Frame $\mathcal{S}$ moves with speed v in the *negative* x direction with respect to inertial system $\mathcal{S}_0$, (i.e. $\boldsymbol{v} = -v\hat{x}$). Their origins coincide at $t = 0$.

a. [0.5 points] Write the Lorentz transformation matrix relating frame $\mathcal{S}$ to frame $\mathcal{S}_0$, in terms of $\beta = v/c$ and $\gamma = 1/\sqrt{1-v^2/c^2}$. [Be careful with the direction of the velocity]

b. [1.5 points] Apply the matrix from the previous question to the four-vector potential $(V_0/c, \boldsymbol{A}_0)$ and find the vector potential in system $\mathcal{S}$, $(V/c, \boldsymbol{A})$. Now focus on the electric potential, V, and express its components in terms of the coordinates of $\mathcal{S}$. As a result you should get that the electric potential V of a magnetic dipole that moves with constant velocity v along the x-axis (i.e. $\boldsymbol{v} = v\hat{x}$) and passes through the origin at $t = 0$ is

$$V = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{v}\cdot(\boldsymbol{m}\times\boldsymbol{R})\,(1-\beta^2)}{c^2\left[(x-vt)^2+(1-\beta^2)(y^2+z^2)\right]^{3/2}}\,. \tag{2}$$

where $\boldsymbol{R} = (\gamma(x-vt), y, z)$.

c. [1 point] Take the non relativistic limit ($v^2 \ll c^2$) in the expression for the electric potential in system $\mathcal{S}$. Using the rule $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})$ simplify its expression. You should get an expression characteristic of the electric potential of an *electric* dipole $\boldsymbol{p}$; identify $\boldsymbol{p}$ in terms of $\boldsymbol{m}$ and $\boldsymbol{v}$.

d. [*bonus question valid 1 point, only after you have reached 6/10*]

Using $\boldsymbol{E}_0 = -\nabla V_0 - \frac{\partial \boldsymbol{A}_0}{\partial t_0}$ and $\boldsymbol{B}_0 = \nabla\times\boldsymbol{A}_0$ find the electric and magnetic field in system $\mathcal{S}_0$ (where ∇ here is meant in terms of the coordinates of $\mathcal{S}_0$).

Problem 3. Field tensor of a capacitor and its Lorentz transformation - [total: 3.5 points]

A parallel plate capacitor is at rest in system $\mathcal{S}$, where it is tilted at an angle ϕ with the x-axis, in the xy plane, and its plates are parallel to the z axis. The plates width is w, the length is l, the separation is d and they carry a charge density of σ and $-\sigma$ as shown in the Figure 1.

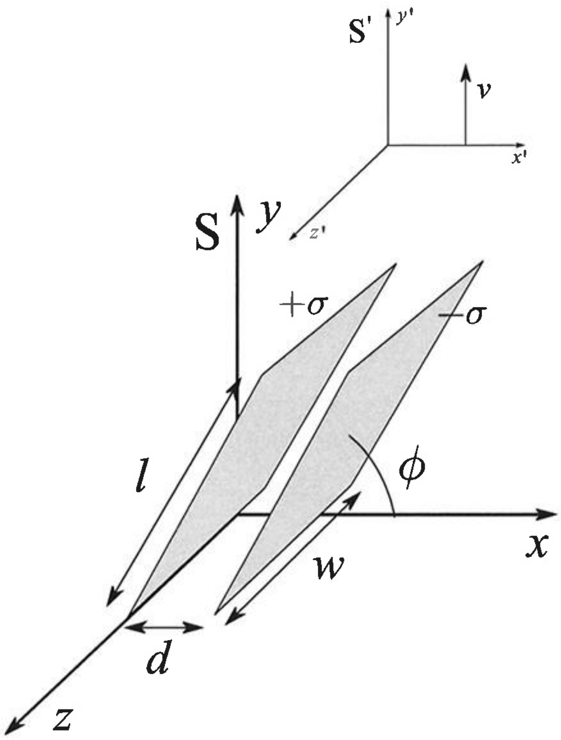

Figure 1: Tilted parallel plate capacitor at rest in system $\mathcal{S}$.

a. [1 point] Write the field tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

generated by the capacitor in system $\mathcal{S}$.

Inertial system $\mathcal{S}'$ moves at constant velocity in the positive y direction, $\boldsymbol{v} = v\,\hat{\boldsymbol{y}}$, with respect to $\mathcal{S}$. Their axes are parallel to one another, and their origins coincide at $t = t' = 0$.

b. [0.5 points] Write the Lorentz transformation matrix relating frame $\mathcal{S}'$ to frame $\mathcal{S}$, in terms of $\beta = v/c$ and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

c. [1 point] Apply the Lorentz transformation to the field tensor $F^{\mu\nu}$ to find $(F')^{\mu\nu}$, the field tensor in $\mathcal{S}'$. [Hint: the transformation law for tensors reads $(F')^{\mu\nu} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}F^{\alpha\beta}$, where Λ^{μ}_{α} is the entry in row μ and column α of the matrix above. Exploit the antisymmetry of the field tensor to minimize calculations.]

Write $\boldsymbol{E}'$ and $\boldsymbol{B}'$.

d. [1 point] Find the length, width and orientation of the plates in the moving frame $\mathcal{S}'$.

Problem 4. Tensor notation and invariant combinations - [total: 2 points]

a. [1 point] The following quantities, constructed solely with the field tensor and its dual tensor, are *invariant* under Lorentz transformation:

- $F^{\mu\nu}F_{\mu\nu}$
- $F^{\mu\nu}G_{\mu\nu}$

Using

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}, \quad G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & -B_z & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

express them in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$. [Remember that when lowering or raising a 0 index, you need to switch the sign, e.g. $F^{02} = -F_{02}$]

Then answer to the following questions:

b. [0.5 points] Can a purely electric field in one inertial system be seen as a purely magnetic field in another?

c. [0.5 points] Can a progressive wave be seen as a purely electric or purely magnetic field in another inertial system?