Answers - MOCK EXAM

1. The distance between emission and the detector is measured in the lab frame. Given that only half of the π^+ mesons reach the detector, it corresponds to the distance traveled by the meson beam in the half-life of the meson as measured in the lab frame. To start, we need to transform the half-life from the rest one, $t^0_{1/2} = 2.5 \cdot 10^{-8}$, to the lab one:

$$t_{1/2} = \gamma\, t^0_{1/2} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} 2.5 \cdot 10^{-8}\,\mathrm{s}\,, \tag{1}$$

and then set the distance traveled in the lab frame equal to this half-life times the velocity of the beam, i.e.

$$\begin{aligned} 15\,\mathrm{m} &= v \cdot t_{1/2} \\ &= \frac{v}{\sqrt{1-\frac{v^2}{c^2}}} 2.5 \cdot 10^{-8}\,\mathrm{s} && (2) \\ &= \frac{v/c}{\sqrt{1-\frac{v^2}{c^2}}} 3 \cdot 10^{8} \cdot 2.5 \cdot 10^{-8}\,\mathrm{s} && (3) \end{aligned}$$

The latter can be easily solved for v/c, after taking the square of the equation. The result is $v = \sqrt{\frac{4}{5}}c$

2. In system S_0 the plates are at rest and making an angle of 30°, equivalently $\pi/6$, with the x_0-axis.

   - The field will be the usual uniform field perpendicular to the plates, pointing toward the negative plate, i.e. making an angle of $-\pi/3$ with the x_0-axis:

   $$\vec{E}_0 = \frac{\sigma_0}{2\epsilon_0}\left(\hat{x}_0 - \sqrt{3}\hat{y}_0\right) \tag{4}$$

   The normal vector perpendicular to the plates and pointing in the direction of the field is $\hat{n}_0 = \frac{1}{2}\left(\hat{x}_0 - \sqrt{3}\hat{y}_0\right)$. So we can also write $\vec{E}_0 = \frac{\sigma_0}{\epsilon_0}\hat{n}_0$. If we call θ_0 the angle that the plates make with the x-axis (in this case 30), and θ_{E_0} the angle that the field makes with the x-axis, we have $\theta_{E_0} = -\left(\frac{\pi}{2} - \theta_0\right)$.

   - The system S moving w.r.t. S_0 along the x-axis to the right at speed v. The transformation of the electric field for a boost in the x-direction gives:

   $$E_x = E^0_x\,, \;\; E_y = \gamma E^0_y\,, \tag{5}$$

   where $\gamma = 1/\sqrt{1 - v^2/c^2}$. Hence we have:

   $$\vec{E} = \frac{\sigma_0}{2\epsilon_0}\left(\hat{x} - \gamma\sqrt{3}\hat{y}\right) \tag{6}$$

- The motion contracts directions parallel to it. The width of the plates will not be affected, since it is in a direction perpendicular to the motion. However the length of the plates, has a component along x and one along y. The component along x is contracted, while the one along y is not. Overall this results in a change of the angle that the plates make with the $x - axis$, specifically this angle will be bigger than the one in system $\mathcal{S}_0$, i.e. θ_0. You can think of the tangent of this angle as given by the ratio of the y-component of the length of the plates, to the x-component of the length of the plates. Hence the tangent of the new angle will be the tangent of the angle in S_0 times γ.

$$\begin{aligned} \tan\theta &= \gamma \tan\theta_0 \\ &= \frac{\gamma}{\sqrt{3}} \end{aligned} \tag{7}$$

  So $\theta = \tan^{-1}(\gamma/\sqrt{3}) > \theta_0$.

- Let us find the angle θ_E that the field $\vec{E}$ makes with the axis:

$$\begin{aligned} \tan\theta_E &= \frac{E_y}{E_x} \\ &= -\gamma\sqrt{3} \end{aligned} \tag{8}$$

  So $\theta_E = \tan^{-1}(-\gamma\sqrt{3})$, corresponding to $\theta_E \neq -\left(\frac{\pi}{2} - \theta\right)$. Hence the field in S is not perpendicular to the plates.

3. $\boldsymbol{E} = E_0\hat{\boldsymbol{x}}$ and $\boldsymbol{B} = \frac{E_0}{2c}(\cos\theta\hat{\boldsymbol{x}} + \sin\theta\hat{\boldsymbol{y}})$

   - In general, given two fields $\boldsymbol{E}$ and $\boldsymbol{B}$ which are not orthogonal, we can transform to a frame in which the new fields will be parallel via a boost in direction $\boldsymbol{E} \times \boldsymbol{B}$ with velocity the solution of the quadratic equation $\beta^2 - b\beta + 1 = 0$, where $\beta = v/c$ and $b = \frac{E^2 + c^2B^2}{\boldsymbol{E}\times\boldsymbol{B}c}$. Hence for this case, we need a frame moving along the $\hat{z}$ direction.

     - Let us find the transformation of the fields for a boost in the $\hat{z}$ direction, with uniform velocity $\bar{v} = v\hat{z}$.

       Using:

$$E'_{\parallel} = E_{\parallel}, \quad \vec{E}'_{\perp} = \gamma\left(\vec{E}_{\perp} + \vec{v}\times\vec{B}\right) \tag{9}$$

$$B'_{\parallel} = B_{\parallel}, \quad \vec{B}'_{\perp} = \gamma\left(\vec{B}_{\perp} - \frac{\vec{v}\times\vec{E}}{c^2}\right) \tag{10}$$

we find

$$
\begin{aligned}
E'_x &= \gamma E_0 \left(1 - \frac{\sin\theta}{2}\frac{v}{c}\right) \\
E'_y &= \gamma E_0 \frac{\cos\theta}{2}\frac{v}{c} \\
B'_x &= \gamma \frac{E_0}{2c}\cos\theta \\
B'_y &= \gamma \frac{E_0}{2c}\left(\sin\theta - 2\frac{v}{c}\right)
\end{aligned} \tag{11}
$$

Therefore the z-component of both fields remains zero, while the x- and y- components change.

– The condition to impose in order for $\boldsymbol{E}' \parallel \boldsymbol{B}'$ is the following

$$
\frac{E'_y}{E'_x} = \frac{B'_y}{B'_x} \tag{12}
$$

Which gives the resulting equation for v/c:

$$
\frac{v^2}{c^2} - \frac{5}{2\sin\theta}\frac{v}{c} + 1 = 0 \tag{13}
$$

The solution for which $v < c$ is:

$$
\frac{v}{c} = \frac{5 - \sqrt{25 - 16\sin^2\theta}}{4\sin\theta} \tag{14}
$$

- Let us now consider the frame $\bar{S}$, which corresponds to a simple rotation of the axis w.r.t S', no boost. In this frame, $\bar{\boldsymbol{E}} = \bar{E}\hat{\bar{\boldsymbol{x}}}$ and $\bar{\boldsymbol{B}} = \bar{B}\hat{\bar{\boldsymbol{x}}}$, with $\bar{E}$ and $\bar{B}$ constant. Therefore the charge will be subject only to an electric force in the $\hat{\bar{\boldsymbol{x}}}$ direction, which will accelerate it but will not change its direction of motion. The magnetic field will not exert any force on the charge, since at all times the velocity of the particle will be parallel to $\bar{\boldsymbol{B}}$. Let us write the velocity of the charge as $\bar{\boldsymbol{v}} = \bar{v}\hat{\bar{\boldsymbol{x}}}$. Newton's second law will therefore read

$$
q\bar{E} = \frac{d}{d\bar{t}}\frac{m\bar{v}}{\sqrt{1 - \frac{\bar{v}^2}{c^2}}} \tag{15}
$$

Given that $\bar{E}$ is constant, we can easily integrate the above equation to get

$$
\frac{m\bar{v}}{\sqrt{1 - \frac{\bar{v}^2}{c^2}}} = q\bar{E}\,t + \text{const} \tag{16}
$$

We now use the initial condition on the velocity to find the constant. Since the initial velocity of the charge is v_0, we have const $= \frac{mv_0}{\sqrt{1-\frac{v_0^2}{c^2}}} \equiv p_0$ (where we use p_0 to indicate the initial value of (the $\hat{\bar{x}}$ component of) the relativistic momentum in the $\bar{S}$ frame; not to be confused with the 0- component of the four vector of the relativistic momentum). Hence we have

$$\frac{m\bar{v}}{\sqrt{1-\frac{\bar{v}^2}{c^2}}} = q\bar{E}\,t + p_0 \tag{17}$$

Squaring the latter and solving for v, we find

$$\bar{v}(\bar{t}) = \frac{\frac{q\bar{E}\bar{t}}{m} + \frac{p_0}{m}}{\sqrt{1+\left(\frac{q\bar{E}\bar{t}+p_0}{mc}\right)^2}} \tag{18}$$

This is the velocity of the charge in the frame $\bar{S}$ along the $\bar{x}$-axis.

- We can finally integrate once more to find the trajectory $\bar{x}(\bar{t})$

$$\begin{aligned} \bar{x}(\bar{t}) &= \int_0^{\bar{t}} \frac{\frac{q\bar{E}t'+p_0}{m}}{\sqrt{1+\left(\frac{q\bar{E}t'+p_0}{mc}\right)^2}} \\ &= \frac{mc^2}{q\bar{E}}\left[\sqrt{1+\left(\frac{q\bar{E}\bar{t}+p_0}{mc}\right)^2} - \sqrt{1+\left(\frac{p_0}{mc}\right)^2}\right] \end{aligned} \tag{19}$$