Final Exam - RED - June 13, 2016 - 10:00-13:00

Problem 1. Particle annihilation and conservation laws [total: 1.5 point]

An electron e^- with kinetic energy of $1\,\mathrm{MeV}$ and rest mass $m = 0.511\,\mathrm{MeV}/c^2$, makes a head-on collision with a positron e^+ at rest. (A positron is an antimatter particle that has the same mass as the electron but opposite charge). In the collision the two particles annihilate each other and are replaced by two photons of equal energy, each traveling at an angle θ with the electron's direction of motion. (A photon is a massless particle with energy $E = pc$). The reaction is

$$e^- + e^+ \to 2\gamma\,, \tag{1}$$

and it is depicted in Figure 1

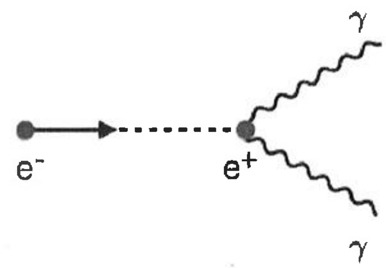

Figure 1: Electron-positron annihilation resulting into two photons.

Use conservation laws to determine:

a. [0.5 points] the relativistic energy E and relativistic momentum p of each photon;

b. [1 points] the angle of emission θ of each photon.

Problem 2. Potentials and Fields of a moving charge - [total: 3.5 points]

A point charge q is at rest at the origin in system $\mathcal{S}_0$.

a. [0.5 points] Considering that in this frame you only have the static Coulomb potential, write the components of the four-vector potential $(V_0/c, \boldsymbol{A}_0)$ associated to this charge (where the 0 indicates that we are in $\mathcal{S}_0$, and does not refer to components).

b. [0.5 points] Now consider a frame $\mathcal{S}$ which moves with velocity v with respect to $\mathcal{S}_0$ along the x-axis in the positive $\hat{x}$ direction. Their origins coincide at $t = 0$. Write the Lorentz transformation matrix relating frame $\mathcal{S}$ to frame $\mathcal{S}_0$, in terms of $\beta = v/c$ and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

c. [1 points] Apply the matrix from the previous question to the four-vector potential $(V_0/c, \boldsymbol{A}_0)$ to find the vector potential in system $\mathcal{S}$, $(V/c, \boldsymbol{A})$. Express V and $\boldsymbol{A}$ in terms of the coordinates of $\mathcal{S}$. As a result you should get that the potentials $V, \boldsymbol{A}$ for a particle that moves with constant velocity v along the x-axis (i.e. $\boldsymbol{v} = -v\hat{x}$) and passes through the origin at $t = 0$, are

$$\begin{aligned} A_x(\boldsymbol{x}, t) &= -\frac{\mu_0}{4\pi} \frac{qv}{[(x + \beta ct)^2 + (1 - \beta^2)(y^2 + z^2)]^{1/2}}, \\ A_y &= A_z = 0, \\ V(\boldsymbol{x}, t) &= \frac{1}{4\pi\epsilon_0} \frac{q}{[(x + \beta ct)^2 + (1 - \beta^2)(y^2 + z^2)]^{1/2}}. \end{aligned} \tag{2}$$

d. [0.5 points] Using

$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}, \quad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}, \tag{3}$$

and eq. (2), find the electric and magnetic field in system $\mathcal{S}$.

e. [1 point] Always in system $\mathcal{S}$, find the Lorentz force exerted on a point charge Q as it passes by the point $\boldsymbol{x} = (0, d, 0)$ of $\mathcal{S}$ at $t = 0$ with velocity $\boldsymbol{u} = u\hat{x}$ as measured in system $\mathcal{S}$. What is the force if $u = -v$?

Problem 3. Field tensor of a capacitor and its Lorentz transformation - [total: 3 points]

A parallel plate capacitor is at rest in system $\mathcal{S}$, where it is lined up with the yz plane. The plates width is w, the length is l, the separation is d and they carry a charge density of σ and $-\sigma$ as shown in the Figure 2.

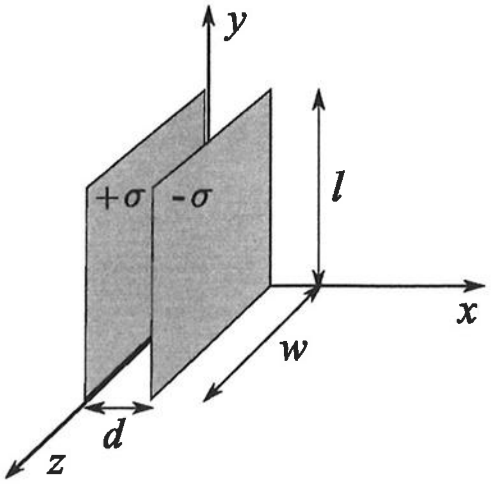

Figure 2: Parallel plate capacitor at rest in system $\mathcal{S}$.

a. [0.5 points] Write the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ field in system $\mathcal{S}$. Use them to construct the field tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

Inertial system $\mathcal{S}'$ moves at constant velocity $\boldsymbol{v} = \beta c(\cos\phi\hat{\boldsymbol{x}} + \sin\phi\hat{\boldsymbol{y}})$ with respect to $\mathcal{S}$.Their axes are parallel to one another, and their origins coincide at $t = t' = 0$. The Lorentz transformation matrix Λ relating such two frames is the following one:

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta\cos\phi & -\gamma\beta\sin\phi & 0 \\ -\gamma\beta\cos\phi & (\gamma\cos^2\phi + \sin^2\phi) & (\gamma-1)\sin\phi\cos\phi & 0 \\ -\gamma\beta\sin\phi & (\gamma-1)\sin\phi\cos\phi & (\gamma\sin^2\phi + \cos^2\phi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

b. [1 point] Apply the Lorentz transformation to the field tensor $F^{\mu\nu}$ to find $F'^{\mu\nu}$, the field tensor in $\mathcal{S}'$. [Hint: the transformation law for tensors reads $F'^{\mu\nu} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}F^{\alpha\beta}$, where Λ^{μ}_{α} is the entry in row μ and column α of the matrix above. Exploit the antisymmetry of the field tensor to minimize calculations.]

c. [0.5 points] Use the above to find the relation between the components of $\boldsymbol{E}'$ and $\boldsymbol{B}'$ and those of $\boldsymbol{E}$ and $\boldsymbol{B}$.

d. [1 point] Find the length and width of the plates in the moving frame $\mathcal{S}'$.

e. [*bonus question valid 1 point, only after you have reached 6/10*] Is the electric field $\boldsymbol{E}'$ perpendicular to the plates of the capacitor in $\mathcal{S}'$? What are the components at $t' = 0$ of the force $\boldsymbol{F}'$ that a test charge would experience if placed inside the capacitor with an initial velocity $\boldsymbol{v} = -v(\cos\phi, \sin\phi, 0)$ with respect to frame $\mathcal{S}'$? In which plane would the motion of this charge be?

Problem 4. Tensor notation and gauge freedom - [total: 2 points]

Consider the definition of the field tensor in terms of the four-vector potential

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \,. \tag{4}$$

a. [0.5 point] Show that $F^{\mu\nu}$ remains unchanged if $A^\mu \to A^{\mu\prime} = A^\mu - \partial^\mu \alpha$.

b. [0.5 points] Rewrite Maxwell's equation $\partial_\mu F^{\nu\mu} = -\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$ as an equation for the four-vector potential.

c. [1 point] Choose the Coulomb gauge defined by $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ and rewrite the latter equation in terms of an equation for the potential V and an equation for the vector potential $\boldsymbol{A}$.