SOLUTIONS of Final Exam - RED 2016

Problem 1. Particle annihilation and conservation laws

The incident electron has momentum p along the positive x-axis, a kinetic energy of $1\,\mathrm{MeV}$ and a rest energy equal to $mc^2 = 0.511\,\mathrm{MeV}$. The positron has only rest energy. Hence the total energy of the initial configuration is

$$E = 1\,\mathrm{MeV} + 2 \times 0.511\mathrm{MeV} = 2.022\mathrm{MeV}\,. \tag{1}$$

Relativistic energy is conserved, hence the two photons emerge from the collision each with energy

$$E_\gamma = \frac{E}{2} = 1.011\,\mathrm{MeV}\,. \tag{2}$$

Correspondingly, the momentum of the photons will be $p_\gamma = E_\gamma/c = 1.011\,\mathrm{MeV}/c$. Using conservation of the momentum in the x-direction, and indicating with $\pm\theta$ the angle that the photons make with the x-axis, we get

$$p = 2p_\gamma \cos\theta\,, \tag{3}$$

giving $\theta = \cos^{-1}\left(\frac{p}{2p_\gamma}\right)$, where p is the momentum of the incoming electron. In order to find the angle, we need to determine the momentum of the incoming electron. We know its rest mass and its kinetic energy, that we can combine to get its relativistic energy: $E = 1.511\,\mathrm{MeV}$. Setting it equal to $E^2 = m^2c^4 + p^2c^2$, we get $p = 1.422\,\mathrm{MeV}/c$. Plugging it in into the expression for the angle, we find $\theta = 45.3°$.

Problem 2. Potentials and Fields of a moving charge

In system $\mathcal{S}_0$ the charge is at rest in the origin, therefore it has only a potential V_0 associated to it, no vector potential. Therefore the four-vector potential will be:

$$A_0^\mu = \left(\frac{q}{4\pi\epsilon_0}\frac{q}{\sqrt{x_0^2+y_0^2+z_0^2}}, 0, 0, 0\right) \tag{4}$$

The Lorentz matrix for a boost in the positive x direction reads:

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Applying it to the four-vector potential, we get the four-vector potential in system $\mathcal{S}$:

$$A^\mu = \gamma\frac{V_0}{c}\begin{pmatrix} 1 \\ -\beta \\ 0 \\ 0 \end{pmatrix}$$

which corresponds to, after transforming also the coordinates with the inverse transformation (i.e. $x_0 = \gamma(x+\beta ct), y = y_0, z = z_0$):

$$\begin{aligned} &V = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{(x+\beta ct)^2 + (1-\beta^2)(y^2+z^2)}}\,, \quad A_x = -\frac{\mu_0}{4\pi}\frac{qv}{\sqrt{(x+\beta ct)^2 + (1-\beta^2)(y^2+z^2)}} \\ &A_y = 0\,, \quad A_z = 0\,. \end{aligned} \tag{5}$$

Now, using

$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}\,, \quad \boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}\,, \tag{6}$$

we get:

$$\begin{aligned} &E_x = \frac{1}{4\pi\epsilon_0}\frac{q\,(1-\beta^2)\,(x+\beta ct)}{\left[(x+\beta ct)^2 + (1-\beta^2)(y^2+z^2)\right]^{3/2}}\,, \quad E_y = \frac{1}{4\pi\epsilon_0}\frac{q\,(1-\beta^2)\,y}{\left[(x+\beta ct)^2 + (1-\beta^2)(y^2+z^2)\right]^{3/2}} \\ &E_z = \frac{1}{4\pi\epsilon_0}\frac{q\,(1-\beta^2)\,z}{\left[(x+\beta ct)^2 + (1-\beta^2)(y^2+z^2)\right]^{3/2}} \\ &B_x = 0\,, \quad B_y = \frac{\mu_0}{4\pi}\frac{qv\,(1-\beta^2)\,z}{\left[(x+\beta ct)^2 + (1-\beta^2)(y^2+z^2)\right]^{3/2}} \\ &B_z = -\frac{\mu_0}{4\pi}\frac{qv\,(1-\beta^2)\,y}{\left[(x+\beta ct)^2 + (1-\beta^2)(y^2+z^2)\right]^{3/2}} \end{aligned} \tag{7}$$

Now, the force at time $t = 0$ on a test charge Q passing in that instant by $(0, d, 0)$ with velocity $\boldsymbol{u} = u\hat{x}$, is given by

$$\boldsymbol{F} = \frac{qQ}{4\pi\epsilon_0} \frac{(1-\beta^2)d}{[(1-\beta^2)d^2]^{3/2}} \left(1 + \frac{u}{c}\frac{v}{c}\right) \hat{y} \tag{8}$$

In the case in which $u = -v$, it reduces to:

$$\boldsymbol{F} = \frac{1}{\gamma} \frac{qQ}{4\pi\epsilon_0} \frac{1}{d^2} \hat{y} \tag{9}$$

which is consistent with the result we would find in system $\mathcal{S}_0$, considering that for $u = -v$ the test charge is at rest in system $\mathcal{S}_0$ and the force is perpendicular to the direction of the boost, hence $F_0 = \gamma F = \frac{qQ}{4\pi\epsilon_0} \frac{1}{d^2} \hat{y}$

Problem 3. Field tensor of a capacitor and its Lorentz transformation

In frame $\mathcal{S}$ there is only a uniform electric field along the $\hat{x}$ direction, with magnitude σ/ϵ_0. The corresponding field tensor is

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & 0 & 0 \\ -E_x/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\sigma}{\epsilon_0 c} & 0 & 0 \\ -\frac{\sigma}{\epsilon_0 c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Applying to this field tensor the Lorentz transformation matrix Λ relating the two frames we get:

$$F^{\mu\nu} = \begin{pmatrix} 0 & \left(\cos^2\phi + \gamma \sin^2\phi\right) E_x/c & (1-\gamma)\cos\phi \sin\phi \, E_x/c & 0 \\ -\left(\cos^2\phi + \gamma \sin^2\phi\right) E_x/c & 0 & \gamma\beta \sin\phi \, E_x/c & 0 \\ -(1-\gamma)\cos\phi\sin\phi \, E_x/c & -\gamma\beta\sin\phi \, E_x/c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This corresponds to the following electric and magnetic fields in frame $\mathcal{S}'$:

$$\begin{aligned} \boldsymbol{E}' &= \left[\left(\cos^2\phi + \gamma\sin^2\phi\right) E_x, (1-\gamma)\cos\phi\sin\phi \, E_x, 0\right] \\ \boldsymbol{B}' &= [0, 0, -\gamma\beta\sin\phi \, E_x/c] \end{aligned} \tag{10}$$

The width of the plates is along the z- axis, hence perpendicular to the direction of the boost and will no be affected by it. The length is along y so it will be affected. Specifically, the component of the length parallel to the direction of the boost, $l_{\parallel} = l\cos\phi$ will be contracted by a factor γ while the perpendicular component, $l_{\perp} = l\sin\phi$ will remain the same. Overall we get

$$l' = \frac{\sqrt{\gamma^2 \sin^2\phi + \cos^2\phi}}{\gamma} \, l \, . \tag{11}$$

We can easily see that $l' = l$ if $\phi = 0$ and, trivially, $\gamma = 1$, while $l' = l/\gamma$ if $\phi = 90°$. This is in agreement with what we expect for a stick along the y-axis when you have a boost in the x-direction, no boost or a boost in the y-direction respectively. Notice that the result for the length in the primed frame can be written also as

$$l' = \sqrt{1 - \beta^2 \sin^2\phi} \; l \tag{12}$$

The electric field in the primed frame now has both an x'- and a y'- component. It has $E'_z = 0$, so it will be in the $x'y'$- plane, making an angle θ_E with the x'- axis:

$$\theta_E = \tan^{-1}\left(\frac{E'_y}{E'_x}\right) = \tan^{-1}\left(\frac{(1-\gamma)\cos\phi\sin\phi}{\cos^2\phi + \gamma\sin^2\phi}\right) \tag{13}$$

This angle will be zero only if $\phi = 0°$ or $90°$ (and trivially for $\gamma = 1$); we will get back to discussing these particular cases after looking at the orientation of the plates.

Let us now analyze the direction of the plates. As we already discussed when looking at the Lorentz contraction of w and l, the width is unaffected by the boost, it remains parallel to the z' axis. The length l however, if observed from the $\mathcal{S}'$ system will be in general contracted and also rotated, acquiring a x' component. We can find this by transforming the coordinates of the endpoints of the length. For simplicity let us take the$l$- side of the positively charged plate which at time $t = 0$ has one extremum at $x^{\mu}_{(1)} = (0,0,0,0)$. This transforms to $x'^{\mu}_{(1)} = (0,0,0,0)$ at $t' = 0$. To observe the other extremum at the same time $t' = 0$ in system $\mathcal{S}'$, we need to transform its coordinates at some instant t (i.e. $(ct, 0, l, 0)$) such that the transformed $t' = 0$:

$$\begin{pmatrix} 0 \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda \begin{pmatrix} ct \\ 0 \\ l \\ 0 \end{pmatrix}$$

where Λ is the Lorentz transformation matrix for the boost given at the beginning. This results in the following three equations:

$$0 = -\gamma\beta \sin\phi \, l + \gamma ct \tag{14}$$

$$x' = (\gamma - 1) \sin\phi \cos\phi \, l - \gamma\beta \cos\phi \, ct \tag{15}$$

$$y' = ((\gamma - 1) \sin^2\phi + \cos^2\phi) \, l - \gamma\beta \sin\phi \, ct \tag{16}$$

Solving (14) for t and substituting into (15) and (16), we get the following spatial coordinates for the other extremum of the l-side of the positively charged plate in system $\mathcal{S}'$:

$$\left(\frac{1-\gamma}{\gamma} \sin\phi \cos\phi, 1 + \frac{1-\gamma}{\gamma} \sin^2\phi, 0 \right) \, l \tag{17}$$

We see that unless $\phi = 0°$ or $90°$, or $\gamma = 1$, the plates will have a non-zero x' component. You can also check that the length of this vector is consistent with the contraction result (11).

We can now check whether the field $\boldsymbol{E}'$ will be parallel to the plates, for instance taking the dot product of $\boldsymbol{E}'$ and the vector $\boldsymbol{l}'$ with extrema $(0,0,0)$ and the point given by (17). The result is

$$\frac{1-\gamma^2}{\gamma} \cos\phi \, \sin\phi \, E_x \, l. \tag{18}$$

which is zero only if $\phi = 0°$ or $90°$ (and trivially for $\gamma = 1$); indeed those are boosts along the x- or $y-axis$, which, as we know, leave the field perpendicular to the plates given the configuration at rest. For other values of ϕ, the electric field will not be perpendicular to the plates.

A test charge placed inside the capacitor with initial velocity $\boldsymbol{v} = -v(\cos\phi, \sin\phi, 0)$ w.r.t. $\mathcal{S}'$, will experience the following force at $t = 0$:

$$\boldsymbol{F}' = q\left(\boldsymbol{E}' + \boldsymbol{v} \times \boldsymbol{B}'\right) = \left[\cos^2\phi + \gamma(1+\beta^2)\sin^2\phi, \left(1 - \gamma(1+\beta^2)\right)\cos\phi \sin\phi, 0\right] \, qE_x \tag{19}$$

Considering that the initial velocity $\boldsymbol{v}$ and $\boldsymbol{E}'$ have only components along x' and y', and that $\boldsymbol{B}'$ is along z' (hence $\boldsymbol{v} \times \boldsymbol{B}'$ has components only along x' and y'), the motion of the test charge will be in the x'-y' plane.

Problem 4. Tensor notation and gauge freedom

Consider the definition of the field tensor in terms of the four-vector potential

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \,. \tag{20}$$

If we now send $A^\mu \to A^{\mu\prime} = A^\mu - \partial^\mu \alpha$, the new field tensor will be:

$$\begin{aligned} F'^{\mu\nu} &= \partial^\mu A'^\nu - \partial^\nu A'^\mu \\ &= \partial^\mu A^\nu - \partial^\mu \partial^\nu \alpha - \partial^\nu A^\mu + \partial^\nu \partial^\mu \alpha \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu \,, \end{aligned} \tag{21}$$

given the symmetry of second derivatives. Inserting the definition of the field tensor in terms of the four-vector potential into Maxwell's equation, we get:

$$\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = -\mu_0 J^\nu \tag{22}$$

Let us rewrite it as:

$$\Box^2 A^\nu - \partial^\nu \partial_\mu A^\mu = -\mu_0 J^\nu \tag{23}$$

The Coulomb gauge corresponds to $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$, therefore to $\partial_\mu A^\mu = \frac{1}{c^2}\frac{\partial V}{\partial t}$. Inserting this in the latter equation, we get:

$$\Box^2 A^\nu - \frac{1}{c^2} \partial^\nu \frac{\partial V}{\partial t} = -\mu_0 J^\nu \tag{24}$$

which choosing $\nu = 0$ and $\nu = i$ (i=1,2,3), can be written as two equations for V and $\boldsymbol{A}$, respectively:

$$\boldsymbol{\nabla}^2 V = -\frac{\rho}{\epsilon_0} \,, \quad \boldsymbol{\nabla}^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} + \frac{1}{c^2} \boldsymbol{\nabla} \frac{\partial V}{\partial t} = -\mu_0 \boldsymbol{J} \,. \tag{25}$$

The equation for $\boldsymbol{A}$ can also be written as:

$$\Box^2 \boldsymbol{A} + \frac{1}{c^2} \boldsymbol{\nabla} \frac{\partial V}{\partial t} = -\mu_0 \boldsymbol{J} \,. \tag{26}$$

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