Exam stf2 Fall/Winter 2011

State clearly your name and - if available - collegekaartnummer. Tip: Read first all questions, and then start with the ones that are easiest for you. Feel free to answer in Dutch, English or both.

1 Adiabatic curve (4 points)

Many ideal gases satisfy equations of state of the form $pV = Nk_BT$ and $E = KNk_BT$. For instance, for monatomic gas $K = 3/2$ and for diatomic gas $K = 5/2$ (in the absence of vibrational degrees of freedom).

(a) Show that such an ideal gas along adiabatic curves obeys

$$pV^{(K+1)/K} = C_1 \quad \text{and} \quad VE^K = C_2$$

where the C_i are constants. [Hint: An adiabatic curve is a curve of constant entropy S for which $dS = \beta dE + \gamma dV = 0$ with $\beta = 1/(k_BT)$ and $\gamma = \beta p$).]

(b) Sketch an adiabatic and an isothermal curve for monatomic gas in the (p, V)-diagram.

2 Entropy of spin system (6 points)

Consider a system of N non-interacting spins in a magnetic field. The spins can be in two states, spin-up and spin-down, with respect to an externally applied magnetic field B, with corresponding energies $\epsilon = \pm mB$ where m denotes the magnetic moment of the spin (the $+$-sign stands for the spin-up state).

(a) Consider a configuration with n spins up and $N - n$ spins down. What is the energy of this configuration? How many different configurations with this energy exist? What is the entropy of the set of configurations with that energy?

(b) For simplicity replace now all the factorials in the expression of entropy using Stirling's formula $\ln(M!) \approx M \ln M$ (that holds asymptotically for large M). Define by $x \equiv n/N$ the fraction of atoms with spin-up. Show that the entropy in terms of x can be written as

$$S \approx -k_BN\left(x \ln x + (1 - x)\ln(1 - x)\right)$$

(c) Calculate now $S(x)$ for $x = 0$, $x = 1/2$ and $x = 1$. Calculate dS/dx for those 3 values. Use this information to sketch $S(x)$.

(d) Give now an expression for the energy E of the system in terms of the fraction x, and determine the minimal and maximal energy of the system. Sketch now $S(E)$, the entropy as a function of E and the temperature T as a function of E. (You do not have to calculate $S(E)$ and $T(E)$ explicitely in order to sketch them!)

(e) Which unusual phenomenon occurs for $E > 0$? Give at least one reason why this phenomenon does usually not occur.

3 Virial expansion (5 points)

Consider a dilute gas of penetrable spheres with a box-like attraction. Their interaction potential $w(r)$ is given by $+W$ for $0 \leq r < D$, by $-U$ for $D \leq r \leq A$ and by zero otherwise (with W, U, D and A being positive numbers with $D < A$).

(a) Calculate the second virial coefficient

$$B_2 = -\frac{1}{2} \int \left(e^{-\beta w(r)} - 1 \right) d^3 r.$$

(b) Give the condition on the set of values W, U, D and A for which there exist a finite value of β with $B_2 = 0$.

(c) Does such a gas with $B_2 = 0$ behave like an ideal gas?

4 Ferromagnetism (5 points)

Consider a system of spins on a lattice. On each site i sits a spin that can assume the values $s_i = \pm 1$.

(a) Consider first a system of non-interacting spins, each with magnetic moment m, in an external magnetic field B (this is the same system as in Problem 2 above). Its Hamiltonian is given by

$$H(\{s_i\}) = -mB \sum_k s_k.$$

Calculate the canonical partition function Z and the mean magnetization $m \langle s \rangle$ per spin.

(b) Now consider the Ising model in a magnetic field. The Hamiltonian is of the form

$$H(\{s_i\}) = -mB \sum_k s_k - J \sum_{\mathrm{NN}} s_i s_j$$

where the second summation goes over the z nearest neighbors. Write down the simplified mean-field Hamiltonian $H_{\mathrm{MF}}(\{s_i\})$ for this model.

(c) Calculate the partition function for H_{MF} and give the implicite equation for $\langle s \rangle$.