Astronomical Relativity 2017
Retake Exam

January 23, 2018

Note: Mark every sheet you hand in with your name and student number, and number the sheets. The clarity of your solutions will factor significantly into the grades. It is not sufficient to write a few equations. You must define your variables, draw well-labeled figures where appropriate, and explain what you are doing. Use geometrized units ($c = G = 1$) throughout, unless specifically instructed otherwise. Note that the instructions are compulsory, for instance if you are instructed to skip mathematical details, lengthy mathematical calculations will result in no points.

1 Problem 1: Gravity as geometry

(a) (1.0 pt) General relativity describes gravity as geometry. Naively, one would think that this principle could also be applied to other forces, such as the electromagnetic force. Explain why a geometric description is not possible for the electromagnetic force.

(b) (1.0 pt) State the equivalence principle. Explain why the equivalence principle is necessary for a geometric theory of gravity. Explain also why the equivalence principle is wider in its implications than just the simple statement (in Newtonian language) that "inertial and gravitational mass are equal".

2 Problem 2: Orbits around a non-rotating black hole

In class we discussed the general properties of orbits of a particle around a non-rotating black hole. We did this by splitting the equation of motion of the particle into a radial and an angular part. We did not consider the angular part, only the radial part, and this is described by

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r), \tag{2.1}$$

where the effective potential V_{eff} is given by

$$V_{\text{eff}}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3} \tag{2.2}$$

and $\mathcal{E}$ is given by

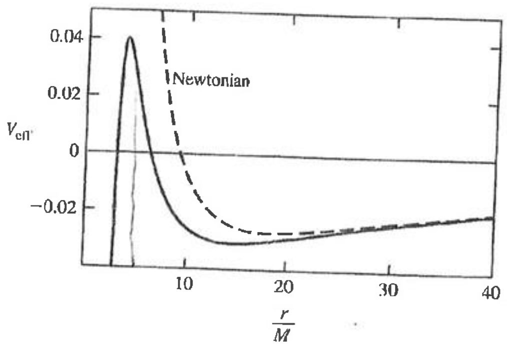
$$\mathcal{E} = \frac{e^2 - 1}{2}. \tag{2.3}$$

In these equations, r is the radial Schwarzschild coordinate, τ is the proper time of the orbiting particle, M is the mass of the black hole, e is the total energy (kinetic + potential energy) of the orbiting particle divided by its rest mass, and ℓ is the orbital angular momentum of the orbiting particle divided by its rest mass. Both e and ℓ are conserved quantities, arising from Killing vectors.

(a) (1.0 pt) Derive expressions for e and ℓ in terms of only Schwarzschild coordinates, proper time, and M.

(b) (1.0 pt) The diagram below (this is actually Fig. 9.2 from Hartle) shows a typical example of the behaviour of the effective potential V_{eff} as a function of Schwarzschild r coordinate. Given V_{eff}, the properties of the orbit of the orbiting particle are determined by the value of $\mathcal{E}$. Now consider the following 4 possible orbits: (i) a stable circular orbit; (ii) a rosette orbit; (iii) a scattering orbit; (iv) a plunge orbit. For each of these 4 cases, make a sketch of the orbit, and of where $\mathcal{E}$ is lying with respect to the effective potential as shown in the diagram (so you need to make 2 sketches for each of the 4 orbits). Do not use any equations.

(c) (1.0 pt) Show that the radius of the innermost stable circular orbit around the black hole is given by $R_{\text{ISCO}} = 6M$.



3 Problem 3: A model FRW Universe

Consider a Friedman-Robertson-Walker model Universe where the geometry is described by the line element

$$ds^2 = -dt^2 + (t/t_0)[dx^2 + dy^2 + dz^2], \tag{3.1}$$

where t_0 is a constant. It obeys the Friedman equation

$$\dot{a}^2 - \frac{8\pi\rho}{3}a^2 = -k, \tag{3.2}$$

where a is the scale factor, ρ is the density, and the constant $k = -1$, 0, or $+1$ depending on the geometry.

(a) (0.5 pt) Is this an open, closed or flat universe?

(b) (0.5 pt) Show that the scale factor in this Universe evolves with time as

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{1}{2}} \tag{3.3}$$

(c) (1.0 pt) Show that the density in this Universe evolves with time as

$$\rho(t) = \frac{3}{32\pi t^2}. \tag{3.4}$$

4 Problem 4: A closed, matter-dominated universe

We know that for much of its past history our Universe was matter-dominated, and this motivates investigating a matter-dominated universe. Here we are going to study a matter-dominated universe with closed geometry. The metric for a closed universe is given on the formula sheet. For a closed universe that is matter-dominated, the scale factor a and the time t can be expressed as a function of a parameter η, as follows:

$$a(\eta) = \frac{\Omega}{2H_0(\Omega - 1)^{3/2}} (1 - \cos\eta) \tag{4.1}$$

$$t(\eta) = \frac{\Omega}{2H_0(\Omega - 1)^{3/2}} (\eta - \sin\eta), \tag{4.2}$$

where the parameter η runs from 0 to 2π.

(a) (0.5 pt) Show that the metric for this universe can be expressed as

$$ds^2 = a^2(\eta)\left[-d\eta^2 + d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)\right]. \tag{4.3}$$

(b) (0.5 pt) Using the metric in this form, η can be used as a time coordinate, and χ as comoving radial spatial coordinate. Draw an η–χ spacetime diagram indicating the Big Bang, Big Crunch, and the time of maximum expansion. Draw also the future light cone of a comoving observer at the moment of the Big Bang, and the past light cone of the same observer at the moment of the Big Crunch. Draw also the past light cone of this observer at the moment of maximum expansion. Explain why you draw the light cones this way.

(c) (0.5 pt) At the moment of the Big Crunch, will this whole spatially finite universe be visible to the observer? Will this also be the case *before* the Big Crunch? If so, when? Explain your answers using the spacetime diagram that you made in question b.

(d) (0.5 pt) In a universe with this geometry, travel along a geodesic would eventually bring the traveller back to the point of departure. Could an observer make this trip in the time available between the Big Bang and the Big Crunch? Derive and explain your answer using the spacetime diagram that you constructed, without complex calculations. (*Hint:* It is helpful to begin by considering (and sketching in your spacetime diagram) the worldline of a light ray emitted at the Big Bang).

Linearized Plane Gravitational Wave

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + h_{\alpha\beta} dx^\alpha dx^\beta$$

where (rows and columns in t, x, y, z order)

$$h_{\alpha\beta}(t, z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_+(t-z) & f_\times(t-z) & 0 \\ 0 & f_\times(t-z) & -f_+(t-z) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for a wave propagating in the z-direction.

Friedman–Robertson–Walker Cosmological Models

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + \left\{\begin{matrix} \sin^2\chi \\ \chi^2 \\ \sinh^2\chi \end{matrix}\right\}(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \quad \left\{\begin{matrix} \text{closed} \\ \text{flat} \\ \text{open} \end{matrix}\right\}.$$

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \quad \begin{pmatrix} k = +1, \text{ closed} \\ k = 0, \text{ flat} \\ k = -1, \text{ open} \end{pmatrix}.$$

THE GEODESIC EQUATION

- Lagrangian for the Geodesic Equation of a test particle

$$L\left(\frac{dx^\alpha}{d\sigma}, x^\alpha\right) = \left(-g_{\alpha\beta}(x)\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\right)^{1/2}$$

  where σ is an arbitrary parameter along the world line $x^\alpha = x^\alpha(\sigma)$ of the geodesic.
- Geodesic equation for a test particle (coordinate basis)

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} \quad \text{or} \quad \frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma$$

  where τ is the proper time along the geodesic and $u^\alpha = dx^\alpha/d\tau$ are the coordinate basis components of the four-velocity so that $\mathbf{u}\cdot\mathbf{u} = -1$. The Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ follow from Lagrange's equations or from the general formula (8.19). The geodesic equation for light rays takes the same form with τ replaced by an affine parameter and $\mathbf{u}\cdot\mathbf{u} = 0$.
- Conserved Quantities

$$\boldsymbol{\xi}\cdot\mathbf{u} = \text{constant}$$

where $\boldsymbol{\xi}$ is a Killing vector, e.g., $\xi^\alpha = (0, 1, 0, 0)$ in a coordinate basis where the metric $g_{\alpha\beta}(x)$ is independent of x^1.

IMPORTANT SPACETIMES (geometrized units)

Flat Spacetime

Cartesian Coordinates

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \equiv \eta_{\alpha\beta}dx^\alpha dx^\beta$$

Spatial Spherical Polar Coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Static, Weak Field Metric

$$ds^2 = -(1 + 2\Phi(x^i))\, dt^2 + (1 - 2\Phi(x^i))(dx^2 + dy^2 + dz^2), \quad (\Phi(x^i) \ll 1).$$

Schwarzschild Geometry

Schwarzschild Coordinates

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Eddington–Finkelstein Coordinates

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Kruskal–Szekeres Coordinates

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} \left(-dV^2 + dU^2\right) + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Kerr Geometry

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mar\sin^2\theta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$
$$+ \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right) \sin^2\theta\, d\phi^2,$$

where

$$a \equiv J/M, \quad \rho^2 \equiv r^2 + a^2\cos^2\theta, \quad \Delta \equiv r^2 - 2Mr + a^2$$