Introduction General Relativity and Astrophysical Applications 2016
Retake Exam

January 30, 2017

Note: Make every problem on a separate sheet. Mark every sheet that you hand in with your name and student number, and number the sheets. As with the problem sets, the clarity of your solutions will factor significantly into the grades. It is not sufficient to write a few equations. You must define your variables, draw well-labeled figures where appropriate, and explain what you are doing. Use geometrized units ($c = G = 1$) throughout, unless specifically instructed otherwise. Note that the instructions are compulsory, for instance if you are instructed to skip mathematical details, lengthy mathematical calculations will result in no points.

1 Problem 1: Gravity and relativity

(a) (0.5 pt) In contrast to the electromagnetic interaction, Newtonian gravity is not consistent with Special Relativity. Explain why this is so.

(b) (0.5 pt) This problem is solved in the description of gravity provided by General Relavity. Explain how.

Solution

(a) Special Relativity implies that no signal can travel faster than the speed of light. The electromagnetic interaction is consistent with this, since every electromagnetic signal (for instance, resulting from a change in the distribution of charge) travels with the speed of light, as can be shown from Maxwell's Equations. Newtonian gravity on the other hand is instantaneous (it has no time dependence such as in Maxwell's Equations), so any gravitational "signal" (for instance, resulting from a change in the mass distribution) must travel with infinite velocity, which violates Special Relativity.

(b) In General Relativity there is also a signal resulting from the *motion* of mass/energy. This could be called a gravitomagnetic effect. As a result, any gravitational signal (for instance, resulting from a change in the mass distribution), travels with a finite speed. It is found from the the Einstein Equation that this speed of propagation is in fact the speed of light. Therefore, in General Relativity, causality violation (i.e., signals traveling faster than the speed of light) does not occur.

2 Problem 2: Circular motion around a non-rotating black hole

In two different spacecrafts, Luke Skywalker and Han Solo are in the vicinity of a non-rotating black hole of mass M. Luke's spacecraft is located at Schwarzschild coordinate $r = 10\,M$, and is in a circular orbit around the black hole. Han Solo's spacecraft is stationary. Once per orbit Luke's spaceship passes close to that of Han Solo. It will be useful to recall that in Schwarzschild coordinates, the angular velocity Ω of a circular orbit around a non-rotating black hole of mass M is given by $\Omega^2 = M/r^3$.

(a) (1.5 pt) How much time elapses on Luke Skywalker's watch between two successive encounters with Han Solo?

(b) (1.5 pt) How much time elapses on Han Solo's clock between successive passages of Luke Skywalker?

Solution

(a) We choose the coordinate system such that Luke's orbit is in the plane $\theta = \pi$. His 4-velocity is $\mathbf{u}$ with $u^\alpha = (u^t, u^r, u^\theta, u^\phi)$. Since the orbit is in a plane of constant θ, we have $u^\theta = 0$, and since it is circular we also have $u^r = 0$. So $u^\alpha = (u^t, 0, 0, u^\phi) = u^t\,(1, 0, 0, u^\phi/u^t)$. Since $u^t = dt/d\tau$ and $u^\phi = d\phi/d\tau$, this becomes $u^\alpha = u^t\,(1, 0, 0, d\phi/dt)$. But $d\phi/dt$ is just the angular velocity in Schwarzschild coordinates, which is Ω. So we get $u^\alpha = u^t\,(1, 0, 0, \Omega)$. The normalization condition $\mathbf{u} \cdot \mathbf{u} = -1$ can now be used to calculate u^t with the result

$$u^t = \left(1 - \frac{2M}{r} - r^2\Omega^2\right)^{-1/2} = \left(1 - \frac{3M}{r}\right)^{-1/2}. \tag{2.1}$$

Inserting $r = 10M$ we obtain $u^t = \sqrt{10/7}$.

We now first look at the period of Luke's orbit measured in Schwarzschild coordinates. In Schwarzschild coordinates, the angular velocity is $\Omega = d\phi/dt$ where t is Schwarzschild time. The period in Schwarzschild coordinate time t is found by using $d\phi = 2\pi$ in this expression, yielding a period $2\pi/\Omega$. Inserting $r = 10M$ in the expression for Ω given in the problem gives $\Omega = 10^{-3/2}M^{-2}$. So the period of the orbit in Schwarzschild time is $2\pi/\Omega = 20\sqrt{10}\pi M^2$.

Now we return to Luke Skywalker for whom we found that $u^t = dt/d\tau = \sqrt{10/7}$, which relates Luke's proper time τ to Schwarzschild coordinate time t. So $\tau = \sqrt{7/10}\,t$. Inserting the period in Schwarzschild time t calculated above, we find a period $20\sqrt{7}\pi M^2$ on Luke Skywalker's clock.

(b) This is analogous to the above calculation except that we now need to calculate Han Solo's 4-velocity $\mathbf{u}$ which is $u^\alpha = u^t(1, 0, 0, 0)$, since he is stationary. The normalization condition $\mathbf{u} \cdot \mathbf{u} = -1$ now becomes

$$u^t = \left(1 - \frac{2M}{r}\right)^{-1/2}. \tag{2.2}$$

Inserting $r = 10M$ we obtain $u^t = \sqrt{5/4}$. So in Han Solo's proper time, the period is $\sqrt{4/5}$ that in Schwarzschild coordinate time, so the period measured by Han Solo becomes $40\sqrt{2}\pi M^2$.

3 Problem 3: Energy released by merging black holes

(a) (0.5 pt) State the area theorem for black holes. Use this theorem to answer the following questions.

(b) (0.5 pt) Suppose 2 non-rotating black holes, each of mass M, merge into a single non-rotating black hole. What it is the maximum amount of energy that can be released in this process?

(c) (1.0 pt) Now consider a small black hole with mass M_1 falling into a much larger black hole with mass M_2. Assume $M_2 \gg M_1$. This situation can be considered as the merging of 2 black holes of very unequal mass. What is the maximum fraction of the mass of M_1 that can be released in the form of energy in this situation? *Hint:* It will be useful to recall that the Taylor expansion of $\sqrt{1+x^2}$ for small x starts with $1+(1/2)x^2$.

Solution

(a) The area of the horizon can never decrease.

(b) The areas of the horizon of the 2 initial black holes are $16\pi M^2$ each, so the total horizon area is $32\pi M^2$. If the mass of the final black hole after merging is M', the corresponding horizon area is $16\pi M'^2$. The area theorem then implies that

$$16\pi M'^2 \geq 32\pi M^2 \tag{3.1}$$

so that $M' \geq \sqrt{2}M$. The energy released in the merging process, which is just $2M - M'$, is therefore at most $(2-\sqrt{2})M$.

(c) Denoting again the mass after the merger by M', the total horizon area before the merger is $16\pi(M_1^2 + M_2^2)$ and after the merger $16\pi M'^2$. The area theorem then implies that $M'^2 \geq M_1^2 + M_2^2$. The energy released in the merging process is $E = M_1 + M_2 - M'$. Combining all this, we obtain

$$E \leq M_1 + M_2 - \sqrt{M_1^2 + M_2^2} = M_2 \left[\frac{M_1}{M_2} + 1 - \sqrt{1 + \left(\frac{M_1}{M_2}\right)^2} \right]. \tag{3.2}$$

Since $M_1 \ll M_2$ we can use a Taylor expanson for the square root. This yields

$$E \leq M_2 \left[\frac{M_1}{M_2} + 1 - 1 - \frac{1}{2}\left(\frac{M_1}{M_2}\right)^2 \right] = M_1 \left(1 - \frac{1}{2}\frac{M_1}{M_2}\right). \tag{3.3}$$

So the fraction of M_1 that can be converted into energy is at most $\frac{1}{2}\frac{M_1}{M_2}$, if $M_1 \ll M_2$.

4 Problem 4: Conditions for a Big Bang

We consider a FRW universe of any geometry (open, closed, or flat), which obeys the Friedman equation

$$\dot{a}^2 - \frac{8\pi\rho}{3}a^2 = -k \tag{4.1}$$

where a is the scale factor. The average density of the universe is ρ (which is the sum of the densities ρ_m, ρ_r and ρ_v of matter, radiation and vacuum). Finally the constant $k = -1$, 0, or

1, depending on the geometry. Radiation and vacuum together produce an internal pressure p which is given by

$$p = \frac{\rho_{\rm r}}{3} - \rho_{\rm v}. \tag{4.2}$$

In this problem you are asked to (eventually) prove that a FRW universe has must have started with a Big Bang if the condition $\rho + 3p > 0$ was always satisfied in the past. This is called the FRW Singularity theorem. The proof is done in three steps.

(a) (0.5 pt) We will use the Friedman equation in dimensionless form. This will involve, among other things, converting the density ρ to a dimensionless density parameter $\Omega = \rho/\rho_{\rm crit}$, where ρ is the critical density. Derive first of all that $\rho_{\rm crit}$ is given by

$$\rho_{\rm crit} = \frac{3H_0^2}{8\pi} \tag{4.3}$$

where H_0 is the Hubble constant.

(b) (1.0 pt) We are now ready to convert the Friedman equation into dimensionless form. Specifically, use $\tilde{a}(t) = a(t)/a_0$, where a_0 is the scale factor at the present time, as the dimensionless scale factor. Use $\tilde{t} = t/t_{\rm H}$, where $t_{\rm H}$ is the Hubble time, as the dimensionless time. Use the density parameter $\Omega = \rho/\rho_{\rm crit}$ as the dimensionless density (split into $\Omega_{\rm m}$, $\Omega_{\rm r}$ and $\Omega_{\rm v}$ for matter, radiation and vacuum). The densities evolve as

$$\rho_{\rm m}(t) = \frac{\rho_{\rm crit}\Omega_{\rm m}}{(\tilde{a}(t))^3} \tag{4.4}$$

$$\rho_{\rm r}(t) = \frac{\rho_{\rm crit}\Omega_{\rm r}}{(\tilde{a}(t))^4} \tag{4.5}$$

Introduce also a density parameter $\Omega_{\rm c}$ for the curvature of the universe, involving only the quantities k, a_0 and H_0. Show that the Friedman equation in these dimensionless parameters can be written as

$$\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 - \Omega_{\rm v}\tilde{a}^2 - \frac{\Omega_{\rm m}}{\tilde{a}} - \frac{\Omega_{\rm r}}{\tilde{a}^2} - \Omega_{\rm c} = 0. \tag{4.6}$$

How do you have to define $\Omega_{\rm c}$ in order for this expression to be correct?

(c) (1.5 pt) Using this dimensionless form of the Friedman equation, now show that the universe must have started with a Big Bang at a finite time in the past, if the condition $\rho + 3p > 0$ was always satisfied. *Hint:* first of all think about what condition you would have to impose on the evolution of $\tilde{a}$ in order to guarantee a Big Bang at a finite time in the past.

Solution

(a) The critical density is the density of a flat universe (i.e., with $k = 0$). The Friedman equation with $k = 0$ reads

$$\dot{a}^2 - \frac{8\pi\rho_{\rm crit}}{3}a^2 = 0. \tag{4.7}$$

Therefore

$$\rho_{\rm crit} = \frac{3}{8\pi}\frac{\dot{a}^2}{a^2}. \tag{4.8}$$

Since $\dot{a}/a$ is the Hubble constant (this does not need to be proved), this yields

$$\rho_{\rm crit} = \frac{3H_0^2}{8\pi}. \tag{4.9}$$

(b) This follows from substitution, which is straightforward as long as they realize that $t_{\rm H} = 1/H_0$. The expression is correct if $\Omega_{\rm c}$ is defined as

$$\Omega_{\rm c} = -\frac{k}{a_0^2 H_0^2}. \tag{4.10}$$

(c) Extrapolating backwards from the present time, a Big Bang means that $a(t)$ crosses zero at a finite time in the past. To *avoid* this, $a(t)$ would have to curve upwards somewhere, meaning $d^2a/dt^2 > 0$. Therefore, if $d^2a/dt^2 < 0$ at all times in the past, a Big Bang *must* have occurred at a finite time in the past. To compute this, we need to differentiate the dimensionless Friedman equation with respect to (dimensionless) time. This yields

$$2\frac{d^2\tilde{a}}{d\tilde{t}^2}\frac{d\tilde{a}}{d\tilde{t}} = 2\Omega_{\rm v}\tilde{a}\frac{d\tilde{a}}{d\tilde{t}} - \frac{\Omega_{\rm m}}{\tilde{a}^2}\frac{d\tilde{a}}{d\tilde{t}} - 2\frac{\Omega_{\rm r}}{\tilde{a}^3}\frac{d\tilde{a}}{d\tilde{t}}, \tag{4.11}$$

so

$$\frac{1}{\tilde{a}}\frac{d^2\tilde{a}}{d\tilde{t}^2} = \Omega_{\rm v} - \frac{\Omega_{\rm m}}{2\tilde{a}^3} - \frac{\Omega_{\rm r}}{\tilde{a}^4}. \tag{4.12}$$

Using $t_{\rm H} = 1/H_0$ this can be converted to

$$\frac{1}{a}\frac{d^2a}{dt^2}\frac{8\pi}{3H_0^2} = \frac{1}{a\rho_{\rm crit}}\frac{d^2a}{dt^2} = \frac{8\pi}{3}\left(\Omega_{\rm v} - \frac{\Omega_{\rm m}}{2\tilde{a}^3} - \frac{\Omega_{\rm r}}{\tilde{a}^4}\right), \tag{4.13}$$

which, using the expressions for the evolution of the density parameter given in the problem, becomes

$$\frac{1}{a}\frac{d^2a}{dt^2} = \frac{4\pi}{3}\left(2\rho_{\rm v} - \rho_{\rm m} - 2\rho_{\rm r}\right). \tag{4.14}$$

Using $p = \rho_{\rm r}/3 - \rho_{\rm v}$, this results in

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi}{3}(\rho + 3p). \tag{4.15}$$

So d^2a/dt^2 is always negative if $\rho + 3p > 0$. In that situation there is always a Big bang at a finite time in the past. This completes the proof.