Master Course in Compact Objects and Accretion (COA) at Leiden University

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The following questions must be answered within 3 hours. In the formulae sheet that accompanies this text, you will find all possible equation that you might need (or not) to answer the questions. Please, note that not all the formulae are needed.

Beside a calculator (if needed) you cannot have on the desk any other study material nor your mobile.

The first 5 questions require a long paragraph answer (so NO "YES or NO" answers, you must give a satisfactory answer with the "whats" and "whys"). The 6th question should instead be a personal summary about one of the general themes treated during the course. This summary may take up to a full page (A4). In this latter, you must put together information learnt along the course in a complete though concise assay with an introduction, main body and conclusions.

The first 5 questions give up to 3 points, the last gives up to 5 points.

Questions:

1. What is the dominant source of pressure in a white dwarf? How does it compare to other sources of pressure ?
2. Why is it important to observationally constrain the maximum mass of a neutron star? How can you measure the mass of a neutron star ?
3. What is the Schwarzchild radius ?
4. What is the role of viscous processes in accretion discs ?
5. Which conditions should be satisfied to have stable mass transfer in a binary ?
6. Please, summarize what you have learnt about supermassive black holes (max a page)?

FORMULAE SHEET

COMPACTNESS $\quad \Theta = \dfrac{GM}{Rc^2}$

ESCAPE VELOCITY $\quad v_{esc} = \sqrt{\dfrac{2GM}{R}} = \sqrt{2\Theta}\, c$

SCHWARZSCHILD RADIUS $\quad R_s = \dfrac{2GM}{c^2}$

FERMI-DIRAC DISTRIBUTION $\quad \dfrac{dn}{dp} = \dfrac{4\pi g}{h^3} \dfrac{p^2}{e^{E-\mu/kT} - 1}$

$\mu \equiv$ chemical potential

$g \equiv 2s+1$ energy level degeneracy ($S \equiv$ spin)

$E^2 = p^2c^2 + m^2c^4$

for $kT >> \mu$: $\dfrac{dn}{dp} = \dfrac{4\pi g}{h^3} p^2 e$ (Maxwell-Boltzmann)

NUMBER DENSITY $\quad n = \displaystyle\int_0^{+\infty} \frac{dn}{dp}\, dp$

ENERGY DENSITY $\quad u = \displaystyle\int_0^{+\infty} \left(E(p) - mc^2 \right) \frac{dn}{dp}\, dp$

PRESSURE $\quad P = \displaystyle\int_0^{+\infty} \frac{1}{3} p v \frac{dn}{dp}\, dp$

where $v = \dfrac{pc^2}{E}$

DEGENERATE GAS OF FERMIONS AT T = 0

$$\frac{dn}{dp} = \frac{4\pi g}{h^3} p^2 \times \begin{cases} 1 & E < \mu \\ 0 & E > \mu \end{cases}$$

FERMI ENERGY $E_F = \mu$

FERMI MOMENTUM $p_F = mc\sqrt{\left(\frac{E_F}{mc^2}\right)^2 - 1}$

FERMI TEMPERATURE $kT_F = E_F - mc^2$

DEGENERATE ELECTRON PRESSURE

$$P = \begin{cases} \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5 m_e m_u^{5/3}} \left(\frac{\rho}{\mu_e}\right)^{5/3} & \text{NON RELATIVISTIC CASE} \\ \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{m_u^{4/3}} \left(\frac{\rho}{\mu_e}\right)^{4/3} & \text{RELATIVISTIC CASE} \end{cases}$$

$\mu_e \equiv$ mean molecular weight per electron

EINSTEIN EQUATION

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ EINSTEIN TENSOR

$T_{\mu\nu} = (\rho c^2 + P) u_\mu u_\nu + P g_{\mu\nu}$ ENERGY-MOMENTUM TENSOR

TOLMAN - OPPENHEIMER - VOLKHOFF of hydrostatic equilibrium

$$\frac{dP}{dr} = \frac{\rho m}{r^2}\left(1 + \frac{P}{\rho}\right)\left(1 + \frac{4\pi P r^3}{m}\right)\left(1 - \frac{2m}{r}\right)^{-1}$$

GRAVITATIONAL REDSHIFT

$$\frac{\lambda_{obs}}{\lambda_0} = 1 + z = \frac{1}{\sqrt{1 - 2\Theta}}$$

FRW METRIC $$-c^2 d\tau^2 = -\left(1-\frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1-\frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi\right)$$

for $r \gg R_S$ $\quad d\tau = dt\left(1-\frac{R_S}{r}\right)^{1/2}$

RADIAL GEODESICS ($d\vartheta = d\varphi = 0$) $$\frac{1}{c}\frac{dr}{dt} = \pm\left|1-\frac{R_S}{r}\right|$$

LSCO for a Schwarzschild BH $\quad R_{LSCO} = 3R_S$, $\quad \ell = \sqrt{3}R_S c$

LSCO for a Kerr BH $\quad R_{LSCO} = 1/2 R_S$ max spin

GRAVITATIONAL RADIUS $\quad R_g = \frac{GM}{c^2}$

EVENT HORIZON RADIUS KERR BH $\quad R_H = R_g + \sqrt{R_g^2 - a}$

where $a = \frac{J}{Mc}$

ACCRETION LUMINOSITY $\quad L_{acc} = \eta \dot{m} c^2$

EDDINGTON LUMINOSITY $\quad L_{Edd} = \frac{4\pi G M m_p c}{\sigma_T} = 10^{38}\, erg/s \left[\frac{M}{M_\odot}\right]$

KEPLER 3rd LAW $$a^3 = \frac{GM_1}{4\pi}(1+q)T^2$$

where $q = \frac{M_2}{M_1}$, $a \equiv$ orbital separation

MASS FUNCTION $$f(M_1, M_2, i) = \frac{(M_2 \sin i)^3}{(M_1+M_2)^2} = \frac{P v_1^3}{2\pi G}$$

where $v_1 = \frac{2\pi}{P} a_1 \sin i$

$i \equiv$ orbital inclination angle

ROCHE POTENTIAL $$\Phi_R(r) = -\frac{GM_1}{|r-r_1|} - \frac{GM_2}{|r-r_2|} - \frac{1}{2}(\omega \times r)^2$$

RADIUS OF SPHERE WITH SAME VOLUME AS THE ROCHE LOBE $$\frac{R_2}{a} = \frac{0.49\, q^{2/3}}{0.6\, q^{2/3} + \ln(1+q^{1/3})}$$

N.B. the lobe radius R_1 is given by replacing q by q^{-1}

ORBITAL ANGULAR MOMENTUM OF A BINARY $$J = M_1 M_2 \left(\frac{G a}{M_1 + M_2}\right)^{1/2}$$

CHANGE IN ORBITAL SEPARATION $$\frac{\dot{a}}{a} = -2\frac{(-\dot{J})}{J} + 2(1-q)\frac{(-\dot{M}_2)}{M_2}$$

CHANGE IN ROCHE LOBE'S SIZE $$\frac{\dot{R}_2}{R_2} \simeq -2\frac{(-\dot{J})}{J} + 2\left(\frac{5}{6} - q\right)\frac{(-\dot{M}_2)}{M_2}$$

$$q_{crit} \cong \frac{5}{6} + \frac{x}{2} - \frac{(-\dot{J})/J}{(-\dot{M}_2)/M_2}$$

LOSS OF ANG. MOMENTUM FOR GW EMISSION $$\left.\frac{(-\dot{J})}{J}\right|_{GW} = \frac{32}{5}\frac{G^3}{c^5}\frac{M_1 M_2 (M_1 + M_2)}{a^4} =$$

$$= \frac{32}{5}(2\pi)^{8/3}\frac{G^{5/3}}{c^5}\frac{M_1 M_2}{(M_1+M_2)^{1/3} P^{8/3}}$$

CATACLYSMIC VARIABLE

MASS-PERIOD RELATION FOR M_2

$$M_2 \simeq \left(4\pi^2\left(\frac{\alpha}{\beta}\right)^3 \frac{1}{G}\right)^{\frac{1}{1-3x}} P^{-\frac{2}{1-3x}}$$

$$M_2 \simeq \begin{cases} 0.11 M_\odot \left(\frac{P}{1h}\right) & \text{MS}, \ x \simeq 1 \\ 0.013 M_\odot \left(\frac{P}{1h}\right)^{-1} & \text{WD}, \ x \simeq -\frac{1}{3} \end{cases}$$

ORBITAL EVOLUTION

$$-\frac{\dot{M}_2}{M_2} \simeq \begin{cases} \frac{1}{\frac{4}{3} - q}\frac{(-\dot{J})}{J} & \text{MS} \\ \frac{1}{\frac{2}{3} - q}\frac{(-\dot{J})}{J} & \text{WD} \end{cases}$$

$$\frac{(-\dot{a})}{a} \simeq \begin{cases} \frac{-2}{4-3q}\frac{(-\dot{J})}{J} & \text{MS} \\ \frac{2}{2-3q}\frac{(-\dot{J})}{J} & \text{WD} \end{cases}$$

X-RAY BINARIES

$$v_{WIND} \simeq \sqrt{\frac{2GM_2}{R_2}}$$

ACCRETION RADIUS $$R_{acc} = \frac{2GM_1}{v_{WIND}^2}$$

ACCRETION RATE $$\dot{M}_{acc} = \frac{1}{4q^2}\left(\frac{R_2}{a}\right)^2 \dot{M}_W$$

where $\dot{M}_W$ mass loss by wind

ACCRETION DISC

HYDROSTATIC EQUILIBRIUM EQ. $\frac{1}{\rho}\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}\left[\frac{GM}{(R^2+z^2)^{1/2}}\right]$

MASS CONSERV. EQ. $R\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial R}(R\Sigma v_R) = 0$

ACCRETION RATE $\dot{M} = 2\pi R\Sigma(R)(-v_R)$

ANG. MOMENTUM CONSERV. EQUATION $R\frac{\partial}{\partial t}(\Sigma R^2\Omega) + \frac{\partial}{\partial R}(R\Sigma v_R R^2\Omega) = \mathcal{G}$

where $\Omega = \sqrt{\frac{GM}{R^3}}$, $\mathcal{G} \equiv$ the effect of the viscous torques

$v_R \simeq -\frac{3}{2}\frac{R}{t_{visc}(R)}$, $t_{visc} \simeq \frac{R^2}{\nu}$

$\nu \equiv \alpha c_s H$ is the kinematic viscosity

$\Sigma(R) = \rho(R)H(R)$

ENERGY LOST IN VISCOUS DISSIPATION $D(R) = \frac{1}{2}\nu\Sigma(R\Omega')^2 = \frac{9GM\dot{M}}{8\pi R^3}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]$

TIME EVOLUTION OF SURF. DENSITY $\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left\{R^{1/2}\frac{\partial}{\partial R}\left(\nu\Sigma R^{1/2}\right)\right\}$

DISC LUMINOSITY $L_{disc} = \frac{GM\dot{M}}{2R_*} = \frac{1}{2}L_{acc}$

TEMPERATURE $T(R) = \left\{\frac{3GM\dot{M}}{8\pi R^3\sigma}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]\right\}^{1/4}$

BB SPECTRUM $I_\nu = \frac{2h\nu^3}{c^2\left[e^{h\nu/kT(R)}-1\right]}$

TOTAL FLUX $F_\nu = \frac{2\pi\cos i}{D^2}\int_{R_*}^{R_{out}} I_\nu R\,dR$

$D \equiv$ distance, $i \equiv$ line of sight inclination