== Retake Exam Advanced Measure Theory ==

Friday 8th July 2022, 13:15-16:15h

Lecturer: dr. S.C. Hille (Mathematical Institute, Leiden University)
Teaching assistant: Esmée Theewis

- The exam is 'open book': students are allowed to use the syllabus, books and lecture notes used during the course, as well as their corrected assignments.
- Argue carefully and write clearly. Put your name and student number clearly visible on each piece of paper that you hand in.

The exam consists of four main exercises.

In any of the questions that follow – unless specifically stated otherwise – S is a general Polish space, equipped with its Borel σ-algebra $\mathcal{B}(S)$, and an admissible metric d has been chosen. $\mathrm{BL}(S,d)$ is the associated space of bounded Lipschitz functions and $\mathcal{M}(S)$ the space of finite signed Borel measures on S. As usual, $\langle \mu, f\rangle = \int_S f\,d\mu$.

1.) [Total: 8pt] Let μ be a signed measure on the measurable space (S,Σ).

(a) [4pt] Prove that for all $E\in\Sigma$,

$$|\mu|(E) = \sup\left\{\sum_{j=1}^n |\mu(E_j)| \;:\; n\in\mathbb{N},\ E_1,\ldots,E_n\in\Sigma \text{ disjoint},\ \bigcup_{j=1}^n E_j = E\right\}$$

(Hint: Show '$\leq$' and '$\geq$' separately. In one, use a Hahn decomposition).

Let $\mathrm{BM}(S)$ be the space of $\mathbb{R}$-valued bounded $(\Sigma,\mathcal{B}(\mathbb{R}))$-measurable functions on S.

(b) [2pt] Prove, that for any $g\in\mathrm{BM}(S)$, $g\geq 0$,

$$\langle \mu^+, g\rangle \;=\; \sup_{f\in\mathrm{BM}(S),\ 0\leq f\leq g} \langle \mu, f\rangle.$$

(c) [2pt] Show that

$$\|\mu\|_{\mathrm{TV}} = \sup_{f\in\mathrm{BM}(S),\ 0\leq f\leq 1}\langle\mu,f\rangle - \inf_{f\in\mathrm{BM}(S),\ 0\leq f\leq 1}\langle\mu,f\rangle.$$

2.) [Total: 12pt] Let λ be the Borel-Lebesgue measure on the interval $[0,1]$ with the Euclidean metric. Consider $f\in L^1([0,1],\lambda)$ such that $\|f\|_{L^1}\neq 0$. Define

$$\mu_f(E) := \int_{[0,1]} \mathbb{1}_E\big((x,x)\big)\, f(x)\,d\lambda(x), \qquad E\in\mathcal{B}(\mathbb{R}^2). \tag{1}$$

(a) [4pt] Prove that equation (1) yields a well-defined signed Radon measure on $\mathbb{R}^2$.

(b) [3pt] Show that $|\mu_f| = \mu_{|f|}$ and $\|\mu_f\|_{\mathrm{TV}} = \int_{[0,1]}|f|\,d\lambda$.

(c) [3pt] Argue that $\Gamma := \{(x,x) : 0 \leq x \leq 1\}$ is a Borel set in $\mathbb{R}^2$ and that μ_f is concentrated on Γ.

(d) [2pt] Can μ_f be absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^2$? Motivate your answer.

3.) [Total: 12pt] Let $\mathcal{M}(S)_{\mathrm{BL}}$ denote the space $\mathcal{M}(S)$ of finite signed Borel measures on S equipped with the $\|\cdot\|^*_{\mathrm{BL},d}$-norm topology. It is known that $\mathcal{D} := \mathrm{span}_{\mathbb{R}}\{\delta_x : x \in S\}$ is dense in $\mathcal{M}(S)_{\mathrm{BL}}$.

(a) [3pt] Show that for $f \in \mathrm{BL}(S,d)$ the linear functional on $\mathcal{M}(S)_{\mathrm{BL}}$ defined by

$$\phi_f(\mu) := \int_S f d\mu \tag{2}$$

is continuous, and that the linear map

$$T : \mathrm{BL}(S,d) \to \mathcal{M}(S)^*_{\mathrm{BL}} : f \mapsto \phi_f \tag{3}$$

is continuous.

Define the linear map $L : \mathcal{M}(S)^*_{\mathrm{BL}} \to C(S)$ by means of

$$L(\psi)(x) := \psi(\delta_x), \qquad x \in S,\ \psi \in \mathcal{M}(S)^*_{\mathrm{BL}}. \tag{4}$$

(b) [3pt] Show that L maps into $\mathrm{BL}(S,d)$, continuously.

(c) [4pt] Show that $LT = \mathrm{Id}_{\mathrm{BL}(S,d)}$ and $TL = \mathrm{Id}_{\mathcal{M}(S)^*_{\mathrm{BL}}}$.
(Hint: Exploit the density of $\mathcal{D}$ in $\mathcal{M}(S)_{\mathrm{BL}}$).

(d) [2pt] Argue that $T : \mathrm{BL}(S,d) \to \mathcal{M}(S)^*_{\mathrm{BL}}$ is a linear isomorphism.

4.) [Total: 8pt] Let $P : \mathcal{M}^+(S) \to \mathcal{M}^+(S)$ be a regular Markov operator with dual operator $U : \mathrm{BM}(S) \to \mathrm{BM}(S)$. Here, $\mathrm{BM}(S)$ denotes the vector space of bounded Borel-measurable functions on S. The space of finite step functions is $\|\cdot\|_\infty$-dense in $\mathrm{BM}(S)$. Define $p(x,\cdot) := P\delta_x \in \mathcal{M}^+(S)$, $x \in S$ and let $\overline{P} : \mathcal{M}(S) \to \mathcal{M}(S)$ be the linear extension of P, given by $\overline{P}\mu := P\mu^+ - P\mu^-$.

(a) [2pt] Show that

$$(P\mu)(E) = \int_S p(x,E)\,\mu(dx) \qquad \text{for all } E \in \mathcal{B}(S).$$

(So P is completely determined by $x \mapsto p(x,\cdot)$).

(b) [4pt] Prove: *if $x \mapsto p(x,\cdot) : S \to \mathcal{M}^+(S)$ is continuous for the $\|\cdot\|_{\mathrm{TV}}$-norm topology on $\mathcal{M}^+(S)$, then U maps $\mathrm{BM}(S)$ into $C_b(S)$.*

(c) [2pt] Show, that if U is a bounded linear map $\mathrm{BL}(S) \to \mathrm{BL}(S)$, then $\overline{P}$ is a bounded linear operator on $\mathcal{M}(S)$ for the norm $\|\cdot\|^*_{\mathrm{BL}}$.

The maximal total number of points that can be obtained is 40. The mark for the exam is computed as max(1, *'total points obtained'*/4)*, rounded to one decimal.*