== Exam Advanced Measure Theory ==

Monday 20th June 2022, 14:15-17:15h

Lecturer: dr. S.C. Hille (Mathematical Institute, Leiden University)
Teaching assistant: Esmée Theewis

- The exam is 'open book': students are allowed to use the syllabus, books and lecture notes used during the course, as well as their corrected assignments.
- Argue carefully and write clearly. Put your name and student number clearly visible on each piece of paper that you hand in.

The exam consists of four main exercises.

In any of the questions that follow – unless specifically stated otherwise – S is a general Polish space, equipped with its Borel σ-algebra $\mathcal{B}(S)$, and an admissible metric d has been chosen. $\mathrm{BL}(S,d)$ is the associated space of bounded Lipschitz functions and $\mathcal{M}(S)$ the space of finite signed Borel measures on S. As usual, $\langle \mu, f\rangle = \int_S f\,d\mu$.

1.) [Total: 7pt] Let $\mu \in \mathcal{M}(S)$ and suppose that

$$\langle \mu, f\rangle \geq 0 \quad \text{for all } f \in \mathrm{BL}(S),\ f \geq 0. \tag{1}$$

(a) [3pt] Show that (1) implies that $\mu(C) \geq 0$ for all $C \subset S$ closed.

(b) [4pt] Prove, using the result from (a), that μ is a positive measure.

2.) [Total: 9pt] Fix a non-zero $\mu \in \mathcal{M}(S)$ and let $h_n \in L^1(S,\mu)$, $n \in \mathbb{N}$. The sequence (h_n) is called *uniformly integrable* if

$$\lim_{M\to\infty} \sup_{n\in\mathbb{N}} \int_S |h_n|\, \mathbb{1}_{\{|h_n|\geq M\}}\, d|\mu| = 0. \tag{2}$$

To each h_n we associate the finite signed Borel measure $\mu_n := h_n d\mu$, via

$$\mu_n(E) := \int_E h_n\, d\mu, \qquad E \in \mathcal{B}(S). \tag{3}$$

You may use without proof that $|\mu_n| = |h_n|\, d|\mu|$.

(a) [3pt] Suppose that for some $1 < p < \infty$, $h_n \in L^p(S,\mu)$ for all n and that $\sup_{n\in\mathbb{N}} \big\| |h_n|^p d\mu \big\|_{\mathrm{TV}} < \infty$. Show that (h_n) is uniformly integrable.

(b) [4pt] Prove: if (h_n) is uniformly integrable, then *(i)* $\sup_{n\in\mathbb{N}} \|\mu_n\|_{\mathrm{TV}} < \infty$ and *(ii)* $\{\mu_n : n \in \mathbb{N}\}$ is uniformly tight.

(c) [2pt] Take $S = \mathbb{R}$ with the Euclidean metric, μ the Borel-Lebesgue measure on $\mathbb{R}$ and $h_n := n\mathbb{1}_{[0,\frac{1}{n}]}$. Show that $\{h_n : n \in \mathbb{N}\}$ is uniformly tight and $\sup_{n\in\mathbb{N}} \|\mu_n\|_{\mathrm{TV}} < \infty$, but (h_n) is not uniformly integrable.

3.) [Total: 10pt] Let (S_i, Σ_i), $i = 1, 2$, be measurable spaces and $\phi : S_1 \to S_2$ a (Σ_1, Σ_2)-measurable map. The push-forward measure of $\mu \in \mathcal{M}(S_1, \Sigma_1)$ under the map ϕ is defined as

$$(\phi\#\mu)(E) := \mu(\phi^{-1}(E)), \qquad E \in \Sigma_2. \tag{4}$$

(a) [2pt] Prove, that for any $\mu \in \mathcal{M}(S_1, \Sigma_1)$:

$$(\phi\#\mu)^+ \leq \phi\#(\mu^+) \quad \text{and} \quad (\phi\#\mu)^- \leq \phi\#(\mu^-).$$

For the remainder of this question assume additionally that ϕ is bijective, with inverse $\phi^{-1} : S_2 \to S_1$ that is (Σ_2, Σ_1)-measurable too.

(b) [2pt] Show, using appropriate Hahn decompositions, that in this setting

$$(\phi\#\mu)^+ = \phi\#(\mu^+) \quad \text{and} \quad (\phi\#\mu)^- = \phi\#(\mu^-).$$

Let $\mu, \nu \in \mathcal{M}(S_1, \Sigma_1)$ be such that $\mu \ll \nu$. Let $h \in L^1(S_1, \Sigma_1, \nu)$ be the Radon-Nikodym derivative of μ with respect to ν.

(c) [3pt] Prove, that $\phi\#\mu \ll \phi\#\nu$.

(d) [3pt] Give an expression for the Radon-Nikodym derivative for $\phi\#\mu$ with respect to $\phi\#\nu$, in terms of h and ϕ. Prove your claim.

4.) [Total: 10pt] Fix an admissible metric d on the Polish space S. For any $f : S \to \mathbb{R}$ bounded measurable (i.e. $f \in \mathrm{BM}(S)$), define the linear map

$$T_f : \mathcal{M}(S) \to \mathcal{M}(S) : \mu \mapsto f\,d\mu, \tag{5}$$

where

$$(f\,d\mu)(E) := \int_E f\,d\mu, \qquad E \in \mathcal{B}(S). \tag{6}$$

Hence $\langle f\,d\mu, g\rangle = \langle \mu, fg\rangle$ for all $\mu \in \mathcal{M}(S)$ and $f, g \in \mathrm{BM}(S)$.

Prove the following, where in each statement $\mathcal{M}(S)$ is equipped with the same indicated topology both in the domain and codomain:

(a) [2pt] If $f \in \mathrm{BM}(S)$, then T_f is continuous for the $\|\cdot\|_{\mathrm{TV}}$-norm topology.

(b) [2pt] If $f \in C_b(S)$, then T_f is continuous for the weak topology on measures.

(c) [3pt] If $f \in \mathrm{BL}(S)$, then T_f is continuous for the $\|\cdot\|_{\mathrm{BL}}^*$-norm topology.

(d) [3pt] Prove the converse to (c): if $f \in \mathrm{BM}(S)$ is such that T_f is a bounded linear operator on $(\mathcal{M}(S), \|\cdot\|_{\mathrm{BL}}^*)$, then $f \in \mathrm{BL}(S)$.

The maximal total number of points that can be obtained is 36. The mark for the exam is computed as ('total points obtained'/4) + 1, rounded to one decimal.