Commutative Algebra: Final Exam

Answer each of the following five questions. You may **not** bring notes or homework solutions with you to the exam, but you **may** use any results from the lecture or proved in homework. Even if you do not solve a part completely, you **may** still use its result in your solutions to subsequent parts and questions. Each question part is worth 1 point for a total of 10 points possible.

1. **Units and Integral Extensions.** Let $A \subseteq B$ be an integral ring extension.

   (a) Let $a \in A$. Show that if a is a unit in B, then a is a unit in A.

   (b) Suppose that A is a field and B is a domain. Show that B is also a field. (Hint: You are not expected to use part (a) to solve this problem.)

2. **Faithful Flatness.** Let R be a ring and M an R-module. Suppose that M is flat, and that for any R-module A, if $M \otimes_R A = 0$ then $A = 0$. (Such a module M is called *faithfully flat.*)

   Let $f : A \to B$ and $g : B \to C$ be two R-module homomorphisms. Show that:

   (a) If $\mathrm{id}_M \otimes f$ is an isomorphism $M \otimes_R A \cong M \otimes_R B$, then f is also an isomorphism.

   (b) If $\mathrm{id}_M \otimes f$ is the zero homomorphism $M \otimes_R A \to M \otimes_R B$, then $f = 0$ as well.

   (c) If $M \otimes_R A \to M \otimes_R B \to M \otimes_R C$ is exact at $M \otimes_R B$ then $A \to B \to C$ is exact at B.

3. **Square Power Series.** Let k be a field of characteristic other than 2. Show that the nonzero squares in $k[[x]]$ are exactly those power series that can be written as $x^{2n} \cdot f$, where n is a natural number and $f \in k[[x]]$ is a power series whose constant term is a nonzero square in k. State any version of Hensel's Lemma that you use.

4. **Reduced Rings and Associated Primes.** We say that a ring R is *reduced* if its only nilpotent element is zero.

   (a) Let R be a reduced ring. Show that every localization of R is also reduced.

   (b) Let R be a reduced local ring with maximal ideal $\mathfrak{m}$. Show that if $\mathfrak{m}$ is an associated prime of R (i.e. $\mathfrak{m} = \mathrm{Ann}_R(r)$ for some $r \in R$), then R is a field.

   (c) Suppose that R is a Noetherian reduced ring. Show that every associated prime of R is a minimal prime of R. (This is a partial converse to the result from lecture that every minimal prime is associated.)

5. **Calculating Dimension.** Calculate the dimension of the local ring $R = \mathbb{C}[[x, y]]/(y^2)$.