Differentiable manifolds II – 2022

Final Exam

Wednesday June 15 2022, 14:15-17:00.
This exam has 3 questions on 2 pages. The total number of points to earn is 45.
The grade for the exam will computed as grade $= 1 + \frac{\text{number of points}}{5}$.
Good luck!

1. Consider the n-sphere in $\mathbb{R}^{n+1}$:

$$\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \|x\|^2 = \sum_{i=1}^{n+1} x_i^2 = 1\}.$$

The space $\mathbb{S}^n \times \mathbb{R}^{n+1}$ has the structure of a (trivial) vector bundle over $\mathbb{S}^n$ with projection map $\Pi : \mathbb{S}^n \times \mathbb{R}^{n+1} \to \mathbb{S}^n$ given by $\Pi(p, v) = p$. Addition and scalar multiplication are defined by

$$(p, v_1) + (p, v_2) = (p, v_1 + v_2), \quad \lambda \cdot (p, v) = (p, \lambda v), \lambda \in \mathbb{R}.$$

Consider the closed embedded submanifold

$$T := \left\{(p, v) \in \mathbb{S}^n \times \mathbb{R}^{n+1} : \langle p, v\rangle_{\mathbb{R}^n} = 0\right\} \subset \mathbb{S}^n \times \mathbb{R}^{n+1},$$

where $\langle p, v\rangle_{\mathbb{R}^{n+1}} := \sum_{i=0}^{n} p_i v_i$ denotes the standard inner product on $\mathbb{R}^{n+1}$.

a. **(5pt)** Prove that T is closed under addition and scalar multiplication in $\mathbb{S}^n \times \mathbb{R}^{n+1}$ and that the restriction $\Pi : T \to \mathbb{S}^n$ is surjective.

Recall that the coordinate projections

$$\mathrm{pr}_i : \mathbb{R}^{n+1} \to \mathbb{R}, \quad x = (x_1, \cdots, x_{n+1}) \mapsto x_i,$$

restrict to functions $\mathrm{pr}_i : \mathbb{S}^n \to \mathbb{R}$ and are elements of $C^\infty(\mathbb{S}^n)$. For $p \in \mathbb{S}^n$ we denote by $T_p\mathbb{S}^n$ the tangent space at p and by

$$T\mathbb{S}^n := \{(p, v_p) : p \in \mathbb{S}^n, v_p \in T_p\mathbb{S}^n\},$$

the tangent bundle of $\mathbb{S}^n$.

b. **(5 pt)** Let $v_p \in T_p\mathbb{S}^n$ be a derivation at p. Show that the map

$$\alpha : T\mathbb{S}^n \to \mathbb{S}^n \times \mathbb{R}^{n+1}, \quad (p, v_p) \mapsto (p, v_p(\mathrm{pr}_1), \cdots, v_p(\mathrm{pr}_{n+1})),$$

maps $T\mathbb{S}^n$ into the subspace T.

We consider $T\mathbb{S}^n$ with its usual vector bundle structure and projection map $\pi : (p, v_p) \mapsto p$.

c. **(5 pt)** Using that the inclusion $\iota : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is a smooth embedding, prove that $\alpha : T\mathbb{S}^n \to T$ is a diffeomorphism and that

$$\Pi \circ \alpha = \pi, \quad \alpha(\lambda(p, v)) = \lambda\alpha(p, v), \quad \alpha((p_1, v_1) + (p_2, v_2)) = \alpha(p_1, v_1) + \alpha(p_2, v_2).$$

Conclude that T admits the structure of a smooth vector bundle over $\mathbb{S}^n$ and that $T \simeq T\mathbb{S}^n$.

2. Let (M, g) be a compact oriented n-dimensional Riemannian manifold with boundary ∂M and volume form ω_g. The volume of M is defined to be $\mathrm{vol}(M) := \int \omega_g$. Recall the *divergence operator* which for a vector field X is defined by $\mathrm{div}(X)\omega_g = \mathrm{d}\beta(X)$. Here $\beta : \mathscr{X}(M) \to \Omega^{n-1}(M)$ is the map

$$\beta(X)(X_1, \cdots, X_{n-1}) = \omega(X, X_1, \cdots, X_{n-1}).$$

For vector fields $X, Y \in \mathscr{X}(M)$ we write $\langle X, Y\rangle_g$ for $g(X, Y) \in C^\infty(M)$. Recall that the Riemannian metric g induces isomorphisms

$$\flat : \mathscr{X}(M) \to \Omega^1(M), \quad \sharp : \Omega^1(M) \to \mathscr{X}(M),$$

determined, for $X \in \mathscr{X}(M)$ and $\omega \in \Omega^1(M)$, by

$$\langle X^\flat, \omega\rangle_g := \omega(X) := \langle X, \omega^\sharp\rangle_g.$$

The gradient of $f \in C^\infty(M)$ is the vector field $\mathrm{grad} f$ defined by

$$\mathrm{grad} f := (\mathrm{d}f)^\sharp, \quad \langle \mathrm{grad} f, X\rangle_g := \mathrm{d}f(X) = Xf, \quad \forall X \in \mathscr{X}(M).$$

a. (**5pt**) Show that for all $f \in C^\infty(M)$ we have the identity

$$\mathrm{div}(fX) = f\mathrm{div}(X) + \langle \mathrm{grad} f, X\rangle_g.$$

The *scalar Laplacian* is the map

$$\Delta : C^\infty(M) \to C^\infty(M), \quad f \mapsto \mathrm{div}(\mathrm{grad} f).$$

As before N denotes the outward unit normal to ∂M.

b. (**5 pt**) Prove *Green's identities* for $f, h \in C^\infty(M)$:

$$\int_M f\Delta h \omega_g + \int_M \langle \mathrm{grad} f, \mathrm{grad} h\rangle_g \omega_g = \int_{\partial M} fN(h)\eta_g$$

and

$$\int_M (f\Delta h - h\Delta f)\omega_g = \int_{\partial M} (fN(h) - hN(f))\eta_g.$$

Here η_g denotes the induced volume form on ∂M and $N(f)$ denotes the function obtained by the action of the vector field N on the function f.

c. (**5 pt**) Let (M, g) be a compact oriented Riemannian manifold with boundary and volume form ω_g. Let N be its outward pointing normal vector field. Prove that

$$\mathrm{vol}(\partial M) = \int_M \mathrm{div}(N)\omega_g$$

Let

$$\mathbb{B}^{n+1} := \left\{x \in \mathbb{R}^{n+1} : \|x\| \leq 1\right\},$$

denote the closed unit ball $\mathbb{B}^{n+1}$ in $\mathbb{R}^{n+1}$. The boundary of $\mathbb{B}^{n+1}$ is the n-sphere $\mathbb{S}^n$ as defined in Exercise 1. We equip $\mathbb{B}^{n+1}$ with the Riemannian metric induced from the Euclidean metric

$$g(X, Y) := \sum_{i=1}^{n+1} X_i Y_i, \quad X, Y \in \mathscr{X}(\mathbb{R}^{n+1}), \quad X = \sum_{i=1}^{n+1} X_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^{n+1} Y_i \frac{\partial}{\partial x_i},$$

on $\mathbb{R}^{n+1}$.

d. (**5 pt**) Using the outward pointing normal $N = \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}$, prove the equality

$$\mathrm{vol}(\mathbb{S}^n) = (n+1)\mathrm{vol}(\mathbb{B}^{n+1})$$

3. Let (M, g) be a compact connected oriented Riemannian manifold with volume form ω_g. Let $[a, b] \subset \mathbb{R}$, $\gamma : [a, b] \to M$ a smooth curve and $\omega \in \Omega^1(M)$ a smooth one form. The line integral of ω along γ is defined as

$$\int_\gamma \omega := \int_a^b \gamma^* \omega.$$

The form ω is conservative if $\int_\gamma \omega = 0$ for all smooth curves $\gamma : [a, b] \to M$ with $\gamma(a) = \gamma(b)$.

a. (**5 pt**) A vector field X is conservative if $X^\flat \in \Omega^1(M)$ is conservative. Prove that X is conservative if and only if there exists a function $f \in C^\infty(M)$ such that $X = \mathrm{grad} f$.

Suppose that (M, g) is an oriented 3-dimensional Riemannian manifold. Define

$$\mathrm{curl} : \mathscr{X}(M) \to \mathscr{X}(M), \mathrm{curl}(X) := \beta^{-1}\mathrm{d}X^\flat.$$

Here $\beta : \mathscr{X}(M) \to \Omega^2(M)$ is as defined in Problem 2.

Recall that the first de Rham cohomology space $H^1_{dR}(M)$ is defined as

$$H^1_{dR}(M) := \{\omega \in \Omega^1(M) : \mathrm{d}\omega = 0\}/\{\mathrm{d}f : f \in C^\infty(M)\}.$$

b. (**5 pt**) Assume that $H^1_{dR}(M) = 0$. Prove that X is conservative if and only if $\mathrm{curl}(X) = 0$.