MEASURE THEORY, MAY 2020

Exam 05/28/2020 (max 100 points)

Question 1 [10 points] Let λ denote Lebesgue measure on $\mathbb{R}$. Let $\mathcal{M}^+(\mathbb{R},\mathcal{B}(\mathbb{R}),\lambda)$ be the space of non-negative measurable funtions on $\mathbb{R}$. Prove or disprove the following statement:
There exist functions $f_1, f_2, \ldots \in \mathcal{M}^+(\mathbb{R},\mathcal{B}(\mathbb{R}),\lambda)$ such that

- $f_n \to f$ pointwise as $n \to \infty$,
- $\int f_n \, d\lambda = \frac{1}{n}$ for all $n \in \mathbb{N}$
- $\int f \, d\lambda = 1$.

Question 2 [30 points] Let $X = (0,1] \cap \mathbb{Q}$ and define

$$\mathcal{I} = \{(a,b] \cap \mathbb{Q} : 0 \le a \le b \le 1,\ a, b \in \mathbb{Q}\}$$
$$\mathcal{A} = \left\{ \bigcup_{i=1}^{n} I_i : n \in \mathbb{N},\ \text{and } I_1, \ldots, I_n \in \mathcal{I} \right\}.$$

(1) Prove that $\mathcal{A}$ is an algebra of subsets of X. Show that if $A \in \mathcal{A}$ and $A \neq \emptyset$ then A has infinitely many elements.
(2) Let $\sigma(\mathcal{A})$ be the sigma algebra generated by $\mathcal{A}$ and let $\mathcal{P}(X)$ be the power set of X. Show that they are equal.

Question 3 [20 points] Let $(X,\mathcal{A},\mu)$ be a measure space and let $f : X \to \bar{\mathbb{R}}$ be measurable. (Here, $\bar{\mathbb{R}}$ stands for $\mathbb{R}$ with $\pm\infty$ included) .

(1) Suppose that the measure μ is finite. Show that the function f is integrable if and only if $\sum_{n \ge 0} \mu(\{|f| > n\}) < \infty$.
(2) Let $p \in [1,\infty)$ and assume that $f \in \mathcal{L}^p(X,\mathcal{A},\mu)$. Prove that the measure μ restricted to $\{x \in X : |f(x)| \neq 0\}$ is σ-finite.

Question 4 [20 points] Let $(X,\mathcal{A},\mu)$ be a measure space and suppose that $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are sequences where $f_n, g_n, f, g \in \mathcal{L}(X,\mathcal{A},\mu)$ for each $n \in \mathbb{N}$ and

i) $|f_n| \le g_n$ for all $n \in \mathbb{N}$ and $f_n \to f$ pointwise as $n \to \infty$,
ii) $g_n \to g$ pointwise as $n \to \infty$ and $\int g_n \, d\mu \to \int g \, d\mu$ as $n \to \infty$.

Prove that $\int f_n \, d\mu \to \int f \, d\mu$ as $n \to \infty$.

Question 5 [20 points] Let $f : [0,1] \to [0,\infty)$ be a positive Borel measurable function and suppose that there exists a number $c \in (0,\infty)$ such that

$$\int_0^1 f(t)^n \, dt = c \text{ for all } n \in \mathbb{N}.$$

Show that there exists $A \in \mathcal{B}([0,1])$ such that $f = \chi_A$ Lebesgue-a.e. Here χ_A is the indicator funtion of A.