Examination for the course on

Random Walks

Teacher: Evgeny Verbitskiy

Thursday, February 1, 2018, 14:00–17:00

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation.
- The use of notes or lecture notes is not allowed.
- There are 8 questions. The total number of points is 100 (per question indicated in boldface). A score of ≥ 55 points is sufficient.

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(1) **[5]** Given two stopping times T_1 and T_2, is

$$T = \min\{T_1, g(T_1, T_2)\}, \text{ where } g(x,y) = \begin{cases} (x+y)/2, & \text{if } x+y \text{ is even,} \\ (x+y+1)/2, & \text{if } x+y \text{ is odd,} \end{cases}$$

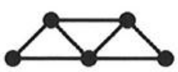
again a stopping time? Prove your answer!

(2) **[10]** Suppose $\{S_n\}$ is the one-dimensional simple random walk. Describe probabilistic properties of the distribution after n-steps $\mathbb{P}(S_n \in \cdot)$. [Exact distribution, limiting behavior as $n \to \infty$, Large deviations].

(2) Suppose $\{S_n^{(d)}\}$ is the d-dimensional simple random walk.

- (a) **[5]** Give definitions of the recurrence and transience of a random walk.
- (b) **[5]** Define the Green function of a random walk and formulate criterion for recurrence in terms of the corresponding Green function.
- (c) **[5]** Give expression of the Green function for the one-dimensional simple random walk.
- (d) **[5]** Sketch the proof of Polya's theorem.

(3) **[5]** Compute the effective resistance between a and b of the following network of unit resistances:

a  b

$\frac{6}{12}$ $\left(\frac{7}{12}+\frac{1}{4}\right)$

$\frac{21}{36}+\frac{4}{36}=\frac{25}{36}$

(4) **[10]** Define the connectivity constant μ for $\mathbb{Z}^2$ and prove that $\mu \in (2,3)$.

(5) (a) **[5]** Formulate the Rayleigh Monotonicity Law for finite networks.

(b) [5] Consider now the infinite network $\mathcal{G}_d = (\mathbb{Z}^d, \mathbb{Z}^d_*)$, $d \geq 1$, where $\mathbb{Z}^d_*$ denotes the set of edges between neighbouring vertices in $\mathbb{Z}^d$. Explain why the effective resistance of $\mathcal{G}_d$ between 0 and infinity is well-defined.

(6) Polymer models.

(a) [5] Recall the probability model for a polymer with the impenetrable substrate (i.e., define the path space $\mathcal{W}_n^+$ and the probability measure $\mathbb{P}_n^{\zeta,+}$ corresponding to the interaction strength ζ).

(b) [5] Define the corresponding free energy $f^+(\zeta)$, and provide expression for $f^+(\zeta)$ in terms of the Green function of a simple random walk.

(7) (a) [5] Suppose $W_t = \sqrt{t} Z_t$ for all $t \geq 0$, where $\{Z_t\}$ are independent Gaussian variables with mean 0 and variance 1. Is $\{W_t\}$ a standard Brownian motion?

(b) [5] Let $(W_t)_{t\geq 0}$ be a standard Brownian motion on $\mathbb{R}$. Is

$$Y(t) = cW_{t/c}$$

again a standard Brownian motion?

(c) [5] Let $(W_t)_{t\geq 0}$ be a standard Brownian motions on $\mathbb{R}$. Show that

$$\operatorname{cov}(W_t, W_s) = \min(s,t) \quad \forall s,t \geq 0.$$

(d) [5] Compute the covariance $\operatorname{cov}(X_t, X_s)$, where $X_t = tW_t$ for all $t \geq 0$, and $\{W_t\}$ is a standard Brownian motion.

(8) Suppose that the current price of a stock is $S_0 = 150$ euro, and that at the end of a single period of time its price is either $S_1 = 100$ euro or $S_1 = 200$ euro. A European call option on the stock is available with a strike price of $K = 155$ euro, expiring at the end of the period. It is also possible to borrow and lend money at a 5% interest rate.

(a) [5] Compute the arbitrage-free price of this option with the help of the Binomial Asset Pricing Model.

(b) [5] Suppose somebody is prepared to buy an option for 0.5 euro more than the arbitrage-free price you have just determined. Explain your action and support your answer by an appropriate calculation.