Examination for the course on

Random Walks

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Wednesday, January 10, 2018, 14:00–17:00

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation.
- The use of notes or lecture notes is not allowed.
- There are 8 questions. The total number of points is 100 (per question indicated in boldface). A score of ≥ 55 points is sufficient.

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(1) Consider simple random walk $(S_n)_{n\in\mathbb{Z}_+}$ on $\mathbb{Z}$.

- (a) **[5]** Give definition of a stopping time.
- (b) **[5]** Provide two examples of non-constant random variables: T_1, which is a stopping time, and T_2, which is not a stopping time. Prove your answer!

(2) Denote by $\{S_n^{(d)}\}$ the simple random walk on the lattice $\mathbb{Z}^d$. Put

$$p_{2n}^{(d)} := \mathbb{P}[S_{2n}^{(d)} = 0], \quad n \in \mathbb{Z}_+.$$

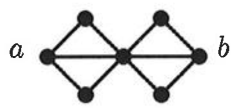
(a) **[10]** For $d = 1, 2$, derive asymptotic estimates of return probabilities $p_{2n}^{(d)}$. Hint: Try to show that $p_{2n}^{(2)} = \left(p_{2n}^{(1)}\right)^2$ for all n directly.

(b) **[5]** Show that the statement "$p_{2n}^{(3)} = \left(p_{2n}^{(1)}\right)^3$ for all n" is false.

(c) **[5]** Define a notion of recurrence of a random walk and formulate a criterion for recurrence.

(d) **[5]** Using the results of (a), show that the simple random walk is recurrent in dimensions $d = 1, 2$.

(3) **[5]** Compute the effective resistance between a and b of the following network of unit resistances:



(4) Let c_n denote the number of self-avoiding walks of length $n \in \mathbb{N}$ on the 'toblerone' graph (i.e., product of $\mathbb{Z}$ and a triangle, in other words 3 copies of $\mathbb{Z}$ that are sideways connected).

(a) [5] Define the connectivity constant μ. State a sufficient condition for the existence of μ?

(b) [5] Compute c_3.

(c) [5] Derive exponential bounds for c_n, and use these bounds to show that $\mu \in (0, \infty)$.

(5) (a) [5] Formulate the Dirichlet Principle.

(b) [5] Formulate the Thomson Principle.

(6) [5] Explain the phenomenon of a phase transition using any of the relevant examples discussed in the course.

(7) Standard Brownian motion.

(a) [5] Sketch construction of a standard Brownian motion $\{W_t\}$ on $[0, 1]$.

(b) [5] Sketch constructions of a standard d-dimensional Brownian motions $\{W_t\}$ on $[0, +\infty)$.

(c) [5] Let $(W(t))_{t\geq 0}$ be a standard Brownian motion on $\mathbb{R}$. Is

$$X_t = W_{3t} - W_{2t}$$

again a standard Brownian motion?

(d) [5] Let $(W(t))_{t\geq 0}$ and $(\widetilde{W}(t))_{t\geq 0}$ be independent standard Brownian motions on $\mathbb{R}$. For which values α and β, is the process

$$\widehat{W_t} = \alpha W_t + \beta \widetilde{W_t},$$

is again a standard Brownian motion.

(8) Suppose that the current price of a stock is $S_0 = 160$ euro, and that at the end of a single period of time its price is either $S_1 = 150$ euro or $S_1 = 175$ euro. A European call option on the stock is available with a strike price of $K = 155$ euro, expiring at the end of the period. It is also possible to borrow and lend money at a 6% interest rate.

(a) [5] Compute the arbitrage-free price of this option with the help of the Binomial Asset Pricing Model.

(b) [5] Suppose somebody is prepared to sell an option for 0.5 euro less than the the arbitrage-free price you have just determined. What is your course of action?